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Spontaneous symmetry breaking, and strings defects in hypercomplex gauge field theories

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Abstract Inspired by the appearance of split-complex structures in the dimensional reduction of string theory, and in the theories emerging as byproducts, we study the hypercomplex formulation of Abelian gauge field theories by incorporating a new complex unit to the usual complex one. The hypercomplex version of the traditional Mexican hat potential associated with the $U(1)$ gauge field theory, corresponds to a *hybrid* potential with two real components, and with $U(1) \times SO(1,1)$ as symmetry group. Each component corresponds to a deformation of the hat potential, with the appearance of a new degenerate vacuum. Hypercomplex electrodynamics will show novel properties, such as spontaneous symmetry breaking scenarios with running masses for the vectorial and scalar Higgs fields, and such as Aharonov–Bohm type strings defects as exact solutions; these topological defects may be detected only by quantum interference of charged particles through gauge invariant loop integrals. In a particular limit, the *hyperbolic* electrodynamics does not admit topological defects associated with continuous symmetries.

1 Introduction

Explorations involving hypercomplex structures have appeared recently in the literature, for example, in the dimensional reduction of M-theory over a Calabi–Yau manifold, where a five-dimensional $\mathcal{N} = 2$ supergravity theory emerges. It turns out that the hyperbolic representation based on para- or split-complex numbers is the most natural way to parametrize the scalar fields of the five-dimensional universal multiplet, gaining insight in the understanding of the string theory landscape [1–6]. In this context, switching on the split-complex form of the theory solves automatically the inconsistencies related with the finding of well-behaved solutions representing the so-called BPS instantons and 3-branes. In the same context, the Lagrangian and the supersymmetric rules used in [7] for a description of the so-called D-instantons in terms of supergravity require by consistency a substitution rule that changes the standard imaginary unit i with $i^2 = -1$, by a formally new imaginary unit j with $j^2 = 1$, which is different from ± 1, and which corresponds to an algebra of para-complex numbers. The formal description of such a mysterious substitution rule is given in [8,9] in terms of para-complex manifolds endowed with a special para-Kähler geometry, with applications in the study of instantons, solitons, and cosmological solutions in supergravity and M-theory.

In a different context, the para-complex numbers appear as a hyperbolic unitary extension of the usual complex phase symmetry of electromagnetism in order to generalize it to a gravito-electromagnetic gauge symmetry [10]. Additionally an alternative representation of relativistic field theories is given in [11–13] in terms of hyperbolic numbers; in particular the Dirac equation and the Maxwell equations admit naturally such a representation, in which one has both the ordinary and the hyperbolic imaginary units; along the same lines it is shown that the $(1+1)$ string world-sheet possesses an inherent hyperbolic complex structure [13]. Furthermore, in [14] the requirement of hermiticity on the Poincaré mass operator defined on the commutative ring of the hyperbolic numbers $\mathcal{H}$ leads to a decomposition of the corresponding hyperbolic Hilbert space into a direct product of the Lorentz group related to the space-time symmetries and the hyperbolic unitary group $SU(4,H)$, which is considered as an internal symmetry of the relativistic quantum state; the hyperbolic unitary group is equivalent to the group $SU(4,C) \times SU(4,C)$ of the Pati–Salam model [15]. In [16] the hyperbolic Klein–Gordon equation for fermions and bosons is considered as a para-complex extension of groups and algebras formulated in terms of the product of ordinary complex and hyperbolic units; this implies the exis-

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tence of hyperbolic complex gauge transformations and the possibility of new interactions; however, although certainly there are no current experimental indications of them, either evidence against their existence. If these new interactions are effectively absent, then it is of interest to understand why the hyperbolic complex counterparts of the other interactions do not exist in nature at presently known energies, in spite of the consistency of the hyperbolic extensions from the theoretical point of view. However, there exists the interesting possibility of realizations of those hyperbolic counterparts beyond presently known energies, including those close to the Planck scale.

On the other hand, the presence of hyperbolic phases implies symmetries associated with non-compact gauge groups; these symmetries can be realized as symmetries of the background space-time and as internal gauge symmetries; for instance, within the context of gravity, it is well known the appearance of non-compact internal symmetries, related to its invariance under diffeomorphisms as the fundamental gauge symmetry. Furthermore, although gauge theories are typically discussed for compact gauge groups, the integrable sectors of QDC, ghost-, and θ-sectors manifest the presence of non-compact gauge groups [17], with surprising new features. For example, the spontaneous symmetry breaking is possible in low dimensions provided that non-compact groups are present, evading the Mermin–Wagner theorem ([18], and references therein); at the quantum level, the Hilbert space is non-separable. Similarly within the study of quantum non-compact σ models, as opposed to the case of quantum electrodynamics, the theory can be correctly quantized only in a Hilbert space with indefinite metric [19]; in the case of a positive-definite Hilbert space, the quantization requires an extended space that incorporates negative-energy modes [20]. In general the classical and quantum descriptions of non-compact σ models show problems related to the non-unitarity of the S matrix and the spontaneous symmetry breaking realizations.

2 Motivations and an advance of results

One of the motivations of the present work is to explore the realizations of the hyperbolic symmetries as an internal gauge symmetry in classical gauge field theories; we determine the effects of the incorporation of those symmetries on the geometry and topology of the vacuum manifolds, and the subsequent effect on the formation of topological defects. We shall find that by switching on the hyperbolic structures, the degenerate vacuum in gauge theories can correspond to non-compact manifolds; the non-compact character of the vacuum will have a nontrivial effect on the possible formation of topological defects associated to continuous symmetries through the Kibble mechanism, when the new hyperbolic symmetry breaks down. Conveniently interpreted, these results are consistent with the lack of strong and convincing evidence of the existence of cosmic topological defects, and consequently, with the possible presence of the hyperbolic symmetry at some stage in the sequence of symmetry breakdowns in the early-universe phase transitions. Incidentally the incorporation of non-compact gauge groups will allow us to gain insight in certain aspects of gravity theories from the perspective of a deformed version of conventional gauge theories.

Topological structures have been object of intense research due to their relevance in confinement and chiral symmetry breaking phenomena in quantum chromodynamics; it is well known the role that the monopoles may play as a possible source of confinement [21,22]; more recently the so-called dyons have been considered as an alternative source for such a phenomenon [23]. In general the topological structure of any gauge theory is conditioned by the existence of nontrivial homotopy groups, and these are determined by the specific gauge symmetry groups and their stability subgroups; recently the Weyl symmetric structure of the classical QCD vacuum has been described by a second homotopy group constraint, which determines the monopole charge [24]. Similarly, the knot topology of QCD vacuum is determined by a third homotopy group constraint [25]; this approach has been useful in the construction of new analytic solutions. With these motivations, in this work we determine the effects of the incorporation of hyperbolic rotations as part of the internal gauge group that usually involves only compact gauge groups, on the topological structures of the considered theories.

In Sect. 3 we consider the hypercomplex numbers with the purpose of introducing the hyperbolic phases as part of the gauge symmetry group, which in general will include the usual $U(1)$ compact phases. Then the hypercomplex deformation with global phases of the classical massive $\lambda\phi^4$ model is developed in Sect. 4; in particular in Sect. 4.1 the purely hyperbolic version of the traditional mexican hat potential is analyzed; hyperbolic version means the substitution of the usual complex unit i by j. This theory does not allow topological defects associated with continuous symmetries. In Sects. 4.2 and 4.3 the hyperbolic deformations of the Mexican hat potential is developed; deformation will imply the incorporation of the new complex unit j, to the usual unit i. In Sect. 4.4 we describe geometrical and topologically the vacuum manifold, which will correspond to a two-dimensional non-compact space embedded in the four-dimensional hypercomplex space as ambient space; the homotopy constraints are analyzed. The polar description of the spontaneous symmetry breakdown is developed in Sect. 4.5; the polar parametrization for the fields will allow us to describe circular, hyperbolic, and radial oscillations, and we shall make a comparison with the usual treatments that involve only compact gauge groups. The formation of possible topological defects are analyzed in Sect. 4.6; in this section Derrick's theorem is confirmed. Furthermore, in the case of theories with $U(1)$ gauge symmetries, the conventional vacuum manifold is defined by the bottom of the *Mexican hat* potential, the circle; in the formulation at hand that circle will be retained as a compact transversal section of the new non-compact vacuum manifold that incorporates the hyperbolic phases.

Finally in Sect. 5 the circular and hyperbolic local rotations are considered by the coupling to hypercomplex electrodynamics; this theory leads to spontaneous symmetry breaking scenarios with hypercomplex scalar and vectorial fields with running masses which mimic the flows of the renormalization groups. In particular, in Sect. 5.3 a purely massive electrodynamics, without the presence of scalar Higgs fields is obtained. In Sect. 5.4 the local topological strings are studied; these defects turn out to be of Aharonov–Bohm type, detectable only by quantum interference of charged particles, in consistency with previous studies on the subject; this issue is discussed in detail in Sect. 6.2 in our concluding remarks. We speculate at the end on possible cosmological implications of this formalism.

3 Incorporating the hyperbolic rotations

As an extension of the conventional complex numbers, the commutative ring of hypercomplex numbers, $z \in \mathcal{H}$, is defined by [10],

$$z = x + iy + jv + ijw, \quad \bar{z} = x - iy - jv + ijw, \quad x, y, v, w \in \mathcal{R} \tag{1}$$

where the hyperbolic unit j has the properties $j^2 = 1$, and $\bar{j} = -j$, and, as usual, $i^2 = -1$, and $\bar{i} = -i$. Hence, with respect to the conjugation involving both complex units, the square of the hypercomplex number is given by

$$z\bar{z} = x^2 + y^2 - v^2 - w^2 + 2ij(xw - yv), \tag{2}$$

which is not a real number, instead it is in general a *Hermitian* number. Equation (2) is invariant under the usual circular rotations $e^{i\theta}$ represented by the Lie group $U(1)$; similarly it is invariant under *hyperbolic* rotations that can be represented by the connected component of the Lie group $SO(1,1)$ containing the group unit. A hyperbolic rotation is represented by the hyperbolic versor $e^{j\chi} \equiv \cosh\chi + j\sinh\chi$, with the split-complex conjugate $e^{-j\chi} = \cosh\chi - j\sinh\chi$, and with the operations $e^{j\chi} \cdot e^{j\chi'} = e^{j(\chi+\chi')}$.

The hyperbolic rotations correspond to a subgroup of the $SL(2,R)$ group, which represents all linear transformations of the plane that preserve oriented area. The elements of the group are classified as elliptic, parabolic, or hyperbolic, depending on whether the $|trace| < 2$, $|trace| = 2$, or $|trace| > 2$, respectively. The respective subgroups are obtained by incorporating $\pm I$, with I the identity element; in particular, the elements of the hyperbolic subgroup are identified with *squeeze* mappings, which geometrically correspond to preserving hyperbolas in the plane, with the hyperbolic angle playing the role of invariant measure of the subgroup. Since the image points of the squeeze mapping are on the same hyperbola, such a mapping preserves the form $x \cdot y$, and can be identified with a *hyperbolic rotation* in analogy with circular rotations preserving circles. The hyperbolic subgroup will play a central role in this paper, since the invariance under its action will be revealed as a fundamental internal symmetry in field theory, due to the fact that the invariant form $x \cdot y$ appears recurrently in physics.

If h is a hyperbolic element, then $|trace(h)| > 2$, and $det(h) = 1$, and can be parametrized as

$$h = \begin{pmatrix} \eta e^{j\chi} & 0 \\ 0 & \eta e^{-j\chi} \end{pmatrix}, \qquad \eta = \pm 1, \qquad \chi \in R - \{0\}; \tag{3}$$

the identity element can be incorporated by allowing $\chi = 0$, and with the choice $\eta = 1$; similarly with $\eta = -1$ the element $-I$ is added. Hence, Eq. (3) parametrizes the elements of the hyperbolic subgroup with $\chi \in R$; the part connected to the identity will be denoted by $SO^+(1,1)$, and represents the subgroup of continuous transformations; the discrete transformation related to the element $-I$ will act separately as a $\mathcal{PT}$-like transformation. Note that the hyperbolic subgroup is an Abelian group, and $SO^+(1,1)$ corresponds to an one-parameter Lie group, with a non-compact generator. Therefore, the quadratic form (2) is invariant under the full phase $e^{i\theta}e^{j\chi}$, corresponding to the group $U(1) \times SO(1,1)$. This symmetry largely ignored in the literature, will have a direct impact in various directions, in particular in the vacuum structure of theories with gauge symmetry, leading to radical changes in its topology and geometry.

The full non-compact group $SL(2,R)$ already has been considered previously as a structure group in a toy model; namely the kinematics of the $SL(2,R)$ "Yang–Mills" theory in 1 + 1 dimensions was studied in [26]; such an analysis was motivated in part by gaining insight in the formulation of gauge theories with non-compact structural groups, which may shed light in the quantization of any theory of gravity. The analysis showed that the configuration space has a non-Hausdorff "network" topology, rather than a conventional manifold, and that the emergent quantization ambiguity cannot be resolved as opposed to the usual compact case. This toy model captures the relevant aspects of a four-dimensional non-compact Yang–Mills theory, which is physically closer to four-dimensional gravity. In that analysis the foliation of the $SL(2,R)$ group by its conjugacy classes is used; the space of conjugacy classes associated with the elliptic and hyperbolic subgroups has the non-compact topology of a two-sheet hyperboloid. A two-sheet hyperboloid has a circle and a hyperbola as factor spaces, with trivial homotopy groups, except $\pi_0 = 2$; in the present work the vacuum manifold will have the circle and the hyperbola as factor spaces, but the homotopy groups will be nontrivial.

We need to restrict ourselves to hypercomplex numbers that have only two degrees of freedom, i.e., that are defined only in terms of two real quantities, in order to obtain a minimal deformation of an ordinary complex number that encodes two real quantities. Such a deformed version will incorporate the hyperbolic rotations to the usual $U(1)$-circular rotations in field theory; this requires the mutual identification of the four real variables in Eq. (1). To this aim, the appropriate identification is $x = \gamma w$ and $y = \gamma v$, with γ a real parameter, which reduces Eq. (1) to

$$
\begin{aligned}
&z = (\gamma + ij)w + (i\gamma + j)v, \quad \bar{z} = (\gamma + ij)w - (i\gamma + j)v,\\
&z\bar{z} = (\gamma^2 - 1)(v^2 + w^2) + 2ij\gamma(w^2 - v^2);
\end{aligned} \tag{4}
$$

hence, the norm is invariant under the interchange of the field $v \leftrightarrow w$, and simultaneously the change $\gamma \to -\gamma$. The effect of a combined circular and hyperbolic rotation is

$$
\begin{aligned}
e^{i\theta}e^{j\chi}z = {} & (\gamma\cos\theta\cosh\chi - \sin\theta\sinh\chi)w\\
&+(\cos\theta\sinh\chi - \gamma\sin\theta\cosh\chi)v\\
&+i[(\gamma\cos\theta\cosh\chi + \sin\theta\sinh\chi)v\\
&+(\gamma\sin\theta\cosh\chi + \cos\theta\sinh\chi)w]\\
&+j[(\cos\theta\cosh\chi - \gamma\sin\theta\sinh\chi)v\\
&+(\gamma\cos\theta\sinh\chi - \sin\theta\cosh\chi)w]\\
&+ij[(\sin\theta\cosh\chi + \gamma\cos\theta\sinh\chi)v\\
&+(\cos\theta\cosh\chi + \gamma\sin\theta\sinh\chi)w];
\end{aligned} \tag{5}
$$

if $\gamma = 0$, then $z = j(v + iw)$, which is essentially an ordinary complex number; however, if $\gamma \neq 0$, then, as opposed to the cases previously considered, the invariant norm in (4) contains necessarily a ij-hybrid term, and it cannot be a purely real quantity. However, the norm can be a purely hybrid quantity by choosing $\gamma^2 = 1$; note that in this case the hypercomplex number in (4) can be reduced to a number proportional to a purely hyperbolic one, and the circular rotation is spurious. On the other hand, the case $\gamma^2 \neq 1$, represents a nontrivial combined rotation (5), and thus a nontrivial algebraic deformation of the usual complex formalism; such a deformation will have a profound effect on the dynamical aspects of the field theories considered.

Furthermore, note that the first term proportional to w in the hypercomplex number in Eq. (4) does not change under conjugation, behaving as the real part of an ordinary complex number; similarly the term proportional to v will have a global change under conjugation, behaving as the imaginary part of an ordinary complex number; thus, one can reverse the expression (4) as

$$
w = \frac{\gamma - ij}{2(\gamma^2 + 1)}(z + \bar{z}), \qquad v = \frac{j - i\gamma}{2(\gamma^2 + 1)}(z - \bar{z}), \tag{6}
$$

where we have used the inverse expressions,

$$
(\gamma + ij)^{-1} = \frac{\gamma - ij}{\gamma^2 + 1}, \qquad (i\gamma + j)^{-1} = \frac{j - i\gamma}{\gamma^2 + 1}, \tag{7}
$$

which are non-singular in spite of belonging to a ring.

Just as any "not-null" hyperbolic number can be brought into polar form $\rho e^{j\chi}$, where $\rho \in \mathcal{R}$ or $\rho \in j \cdot \mathcal{R}$ depending on whether the norm of the number is strictly positive or strictly negative, respectively, any number of the general form (1), with $|z| \neq 0$ can be written as

$$
z = \rho e^{i\theta} e^{j\chi}, \tag{8}
$$

where $\theta \in (0, 2\pi]$, $\chi \in \mathcal{R}$ and, in general, ρ must be a *Hermitian* number, i.e.,

$$
\rho = \rho_R + ij\rho_H. \tag{9}
$$

Equating the two expressions (Cartesian and polar) for z we obtain the relations

$$
\begin{aligned}
x &= \rho_R\cos\theta\cosh\chi - \rho_H\sin\theta\sinh\chi,\\
y &= \rho_R\sin\theta\cosh\chi - \rho_H\cos\theta\sinh\chi,\\
v &= \rho_R\cos\theta\sinh\chi - \rho_H\sin\theta\cosh\chi,\\
w &= \rho_R\sin\theta\sinh\chi - \rho_H\cos\theta\cosh\chi;
\end{aligned} \tag{10}
$$

which can in principle be inverted to obtain ρ_R, ρ_H, θ, and χ in terms of x, y, v, and w. The explicit expressions turn out to be relatively complicated, but we can straightforwardly arrive at the following implicit formulas:

$$
\begin{aligned}
&\rho_R^2 - \rho_H^2 = x^2 + y^2 - v^2 - w^2, \quad \rho_R\rho_H = xw - yv,\\
&\frac{1}{2}(\rho_R^2 + \rho_H^2)\sinh 2\chi = xv + yw,\\
&\frac{1}{2}(\rho_R^2 + \rho_H^2)\sin 2\theta = xy + vw;
\end{aligned} \tag{11}
$$

that is, given the Cartesian components of z, one can use the first two equations to obtain ρ_R and ρ_H in terms of them, and then insert those values into the latter two to get the phases θ and χ.

In the hypercomplex formulation developed, the real objects such as Lagrangians, vector fields, masses, and coupling parameters will be generalized to Hermitian objects, encoding two real objects. The four real components of a hypercomplex field have been identified to each other by using a real γ-parameter, leading to two real effective variables; hence, the new formulation is constructed as a γ-deformation along a non-compact direction defined by the new complex unit. As an effect of the γ-deformation, the traditional Mexican hat potential will be hallowed out in two points in the valley that defines the degenerate vacuum; such two points represent the new vacuum states. In a limit case, a purely hyperbolic version of the Mexican hat potential will be obtained.

4 Hypercomplex version of the classical model $\lambda\phi^4$: global symmetries

The theory "$\lambda\phi^4$" is not only a pedagogical model; for example the potential $\lambda(H^\dagger H)^2$ has been considered recently for supporting the idea of a quantum origin of the Higgs potential and the electroweak scale [27]; specifically the renormalization group formalism for the corresponding Coleman–Weinberg potential obtained by radiative corrections is developed. Hence, the Coleman–Weinberg symmetry breaking can be understood in terms of the running of the coupling constants. However, although the present formulation is far in spirit from the Coleman–Weinberg dynamical symmetry breaking scheme, the hypercomplex deformation of the $U(1)$ $\lambda\phi^4$ theory will be understood, in certain sense, in terms of the running of the coupling constants, as functions of the parameter γ, which has up to this point, an algebraic origin.

Now we shall show that, properly analytically continued, the $\lambda\phi^4_d$ theory for a massive (real) scalar field ϕ, in a d-dimensional flat background, can be re-formulated on the hypercomplex space; the conventional theory is described by

$$\mathcal{L}(\phi) = \int \mathrm{d}x^d \left(\frac{1}{2}\partial^i\phi\cdot\partial_i\phi - V(\phi)\right),$$
$$V(\phi) = \frac{1}{2}am^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{12}$$

where $m^2 > 0$, and the self-interacting constant λ is assumed to be positive in order to have the energy bounded from below; the action in invariant under the action of the cyclic group $Z_2 = \{+1, -1\}$, manifested though the discrete symmetry $\phi \to -\phi$. The unbroken exact symmetry scenario requires $a = 1$, and the vacuum manifold corresponds to a single point, without homotopy constraints, and hence without possible topological defects. Furthermore, the spontaneous symmetry breaking scenario requires $a = -1$, and the vacuum manifold corresponds to the 0-sphere, $S^0 \sim Z_2$, with the nontrivial homotopy constraint $\pi_0 = 2$, and hence admitting *the domain walls* as possible topological defects.

The hypercomplex version is based on Eqs. (4) and (5), and hence the Lagrangian can be re-interpreted in terms of the two hypercomplexified field variables $(\psi, \overline{\psi})$, with

$$\psi\overline{\psi} = (\gamma^2 - 1)(v^2 + w^2) + 2ij\gamma(w^2 - v^2), \tag{13}$$

which is invariant under the global phases $e^{i\theta}e^{j\chi}$, and under the discrete transformation $(v \leftrightarrow w, \gamma \to -\gamma)$,

$$\mathcal{L}(\psi, \overline{\psi}) = \int \mathrm{d}x^d \left[\frac{1}{2}\partial^i\psi\cdot\partial_i\overline{\psi} - V(\psi, \overline{\psi})\right],$$
$$V(\psi, \overline{\psi}) = \frac{a}{2}m^2\psi\overline{\psi} + \frac{\lambda}{4!}\psi^2\overline{\psi}^2, \tag{14}$$

which is *Hermitian* and a non-analytical function on ψ, and hence can attain relative minima and/or maxima; we consider that the square mass is also *Hermitian*, with real and hybrid parts $m^2 \equiv m_R^2 + ijm_H^2$; $a = \pm 1$, and similarly we consider that $\lambda \equiv \lambda_R + ij\lambda_H$, with $(m_R^2, m_H^2, \lambda_R, \lambda_H)$ real parameters. The potential can be written explicitly in terms of its real and hybrid parts as $V = V_R + ijV_H$,

$$\begin{aligned} V_R &= a\left(\frac{\gamma^2-1}{2}m_R^2 + \gamma m_H^2\right)v^2 + a\left(\frac{\gamma^2-1}{2}m_R^2 - \gamma m_H^2\right)w^2 \\ &\quad + \frac{\lambda_R}{6}\left[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2\right] \\ &\quad - \frac{\lambda_H}{6}\gamma(\gamma^2-1)(w^4 - v^4); \end{aligned} \tag{15}$$

$$\begin{aligned} V_H &= a\left(\frac{\gamma^2-1}{2}m_H^2 - \gamma m_R^2\right)v^2 + a\left(\frac{\gamma^2-1}{2}m_H^2 + \gamma m_R^2\right)w^2 \\ &\quad + \frac{\gamma\lambda_R}{6}(\gamma^2-1)(w^4 - v^4) \\ &\quad + \frac{\lambda_H}{6}\left[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2\right]; \end{aligned} \tag{16}$$

one can map the potentials V_R and V_H to each other, by the discrete transformations

$$\gamma \to -\gamma, \quad (\lambda_R, \lambda_H) \to (\lambda_H, \lambda_R), \quad m_R \to m_H. \tag{17}$$

The vacuum is defined as usual by the stationary points constraint,

$$\frac{\partial V}{\partial\psi_0} = \overline{\psi_0}\left[am^2 + \frac{\lambda}{6}\psi_0\overline{\psi_0}\right] = 0; \tag{18}$$

which can be expressed explicitly is terms of real fields (v_0, w_0) and real parameters (m_R, m_H); the zero-energy point for V_R and V_H is described by

$$(v_0 = 0, w_0 = 0)\,; \tag{19}$$

$$\begin{aligned} &V_R = 0; \quad (\det\mathcal{H})_{(V_R)} = 4\left[\left(\frac{\gamma^2-1}{2}\right)^2 m_R^4 - \gamma^2 m_H^4\right]; \\ &\frac{\partial^2 V_R}{\partial v_0^2} = 2a\left(\frac{\gamma^2-1}{2}m_R^2 + \gamma m_H^2\right); \\ &\frac{\partial^2 V_R}{\partial w_0^2} = 2a\left(\frac{\gamma^2-1}{2}m_R^2 - \gamma m_H^2\right); \end{aligned} \tag{20}$$

$$V_H = 0; \qquad \det\mathcal{H}_{(V_H)} = 4\left[\left(\frac{\gamma^2-1}{2}\right)^2 m_H^4 - \gamma^2 m_R^4\right];$$

$$\frac{\partial^2 V_H}{\partial v_0^2} = 2a\left(\frac{\gamma^2-1}{2}m_H^2 - \gamma m_R^2\right),$$
$$\frac{\partial^2 V_H}{\partial w_0^2} = 2a\left(\frac{\gamma^2-1}{2}m_H^2 + \gamma m_R^2\right); \tag{21}$$

where we have displayed the second order derivatives, and the determinant of the Hessian matrix; likewise, the other stationary points for V_R and V_H, related with the condition $am^2 + \frac{\lambda}{6}\psi_0\overline{\psi_0} = 0$, read

$$(1-\gamma^2)(\lambda_R^2+\lambda_H^2)(v_0^2+w_0^2) = 6a(\lambda_R m_R^2 + \lambda_H m_H^2); \tag{22}$$
$$\gamma(\lambda_R^2+\lambda_H^2)(v_0^2-w_0^2) = 3a(\lambda_R m_H^2 - \lambda_H m_R^2); \tag{23}$$

which can be solved to favor of the fields (v_0, w_0):

$$v_0^2 = \frac{3}{2a}\frac{1}{\lambda_R^2+\lambda_H^2}\frac{1}{\gamma(1-\gamma^2)} \times\left\{\left[(\gamma^2-1)\lambda_H + 2\gamma\lambda_R\right]m_R^2 + \left[(1-\gamma^2)\lambda_R + 2\gamma\lambda_H\right]m_H^2\right\},$$
$$w_0^2 = \frac{3}{2a}\frac{1}{\lambda_R^2+\lambda_H^2}\frac{1}{\gamma(1-\gamma^2)} \times\left\{\left[(1-\gamma^2)\lambda_H + 2\gamma\lambda_R\right]m_R^2 - \left[(1-\gamma^2)\lambda_R - 2\gamma\lambda_H\right]m_H^2\right\}, \tag{24}$$
$$V_R = \frac{3}{2}\frac{\lambda_R(m_H^4 - m_R^4) - 2\lambda_H m_H^2 m_R^2}{\lambda_R^2+\lambda_H^2}; \tag{25}$$
$$\frac{\partial^2 V_R}{\partial v_0^2} = \frac{1}{3}\left[(\gamma^4 - 6\gamma^2 + 1)\lambda_R + 4\gamma(\gamma^2-1)\lambda_H\right]v_0^2;$$
$$\frac{\partial^2 V_R}{\partial w_0^2} = \frac{1}{3}\left[(\gamma^4 - 6\gamma^2 + 1)\lambda_R - 4\gamma(\gamma^2-1)\lambda_H\right]w_0^2;$$
$$\left(\frac{\partial^2 V_R}{\partial v_0 \partial w_0}\right)^2 = \left[\frac{\lambda_R}{3}(\gamma^2+1)^2 v_0 w_0\right]^2;$$
$$(\det\mathcal{H})_{(V_R)} = -\left[\frac{4\gamma(\gamma^2-1)}{3}\right]^2(\lambda_R^2+\lambda_H^2)v_0^2 w_0^2; \tag{26}$$
$$V_H = \frac{3}{2}\frac{\lambda_H(m_R^4 - m_H^4) - 2\lambda_R m_H^2 m_R^2}{\lambda_R^2+\lambda_H^2}. \tag{27}$$

Considering that V_H is obtained from V_R by means of the transformations (17), the second order derivatives expressions for V_H can be obtained directly from Eq. (26); in particular $(\det\mathcal{H})_{V_H} = (\det\mathcal{H})_{V_R}$, which is strictly negative, according to the expression (26), and thus the points (24) are in general saddle points for both potentials, unless $\det\mathcal{H} = 0$, then anything is possible. For this purpose, one can fix to zero one of the vacuum expectation values, v_0, or w_0, which in fact will be required by spontaneous symmetry breaking; the choice $w_0 = 0$ leads to the following simplification:

$$w_0 = 0, \quad \rightarrow \quad \frac{m_H^2}{m_R^2} = \frac{(1-\gamma^2)\lambda_H + 2\gamma\lambda_R}{(1-\gamma^2)\lambda_R - 2\gamma\lambda_H}, \tag{28}$$
$$v_0^2 = \frac{6am_R^2}{(1-\gamma^2)\lambda_R - 2\gamma\lambda_H}; \tag{29}$$

fortunately, these conditions will induce two global minima for both potentials V_R and V_H in the case with $\gamma^2 \neq 1$, and with a non-zero vacuum expectation value for the field v; this case will be analyzed in detail in Sect. 4.2. Alternatively one has the choice

$$v_0 = 0, \quad \rightarrow \quad \frac{m_H^2}{m_R^2} = \frac{(1-\gamma^2)\lambda_H - 2\gamma\lambda_R}{(1-\gamma^2)\lambda_R + 2\gamma\lambda_H}, \tag{30}$$
$$w_0^2 = \frac{6am_R^2}{(1-\gamma^2)\lambda_R + 2\gamma\lambda_H}; \tag{31}$$

which can be obtained from (28) and (29) by the change $\gamma \rightarrow -\gamma$; this case will be developed in Sect. 4.3.

Now we expand the theory around ψ_0, using only the degenerate vacuum constraint $\lambda\psi_0\overline{\psi}_0 = -6am^2$,

$$\begin{aligned} V(\psi+\psi_0, \overline{\psi}+\overline{\psi_0}) &= \frac{am^2}{2}(\psi+\psi_0)(\overline{\psi}+\overline{\psi_0}) \\ &+\frac{\lambda}{4!}(\psi+\psi_0)^2(\overline{\psi}+\overline{\psi_0})^2 \\ &= -\frac{am^2}{2}\psi\overline{\psi} + \frac{\lambda}{4!}(\psi\overline{\psi})^2 + \frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2) \\ &+\frac{\lambda}{12}\psi\overline{\psi}(\psi_0\overline{\psi} + \overline{\psi}_0\psi), \end{aligned} \tag{32}$$

note that we have not chosen yet a definite vacuum, but the expansion around a non-zero ground state value for the field leads to a change of sign in the mass term. Furthermore, the first two terms in Eq. (32) have already the canonical form since depend on the norm $\psi\overline{\psi}$ given in (13). The third and fourth terms correspond to the quadratic and cubic terms in the fields (v, w); such terms do not have the canonical form (due to the presence of mixed terms of the form vw) and do not depend on the modulus $\psi_0\overline{\psi}_0$; explicitly we have,

$$\begin{aligned} \overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2 &= 2\cosh 2(\chi_0-\chi)\cos 2(\theta_0-\theta) \\ &\times\Big\{(\gamma^4+1)(w_0^2-v_0^2)(w^2-v^2) - 2\gamma^2 \\ &\times[(v_0^2+3w_0^2)w^2 + (3v_0^2+w_0^2)v^2]\Big\} \\ &+8\gamma\sinh 2(\chi_0-\chi) \\ &\times\Big\{(\gamma^2-1)\sin 2(\theta_0-\theta)(v_0^2v^2 - w_0^2w^2) \\ &-(\gamma^2+1)\cos 2(\theta_0-\theta)v_0w_0(v^2+w^2)\Big\} \\ &+4(\gamma^4-1)\cosh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)v_0w_0(v^2-w^2) \\ &+8(\gamma^2+1)\cos 2(\theta_0-\theta)\Big[(\gamma^2+1)v_0w_0\cosh 2(\chi_0-\chi) \end{aligned}$$

$$
\begin{aligned}
&+\gamma(v_0^2+w_0^2)\sinh 2(\chi_0-\chi)\Big]\underbrace{vw}\\
&-4(\gamma^4-1)(v_0^2-w_0^2)\sinh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)\cdot\underbrace{vw}\\
&+2ij\Big\{4\gamma(\gamma^2-1)\cosh 2(\chi_0-\chi)\cdot\cos 2(\theta_0-\theta)(w_0^2w^2-v_0^2v^2)\\
&+\sinh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)\Big[(\gamma^4+1)(w_0^2-v_0^2)(w^2-v^2)\\
&-2\gamma^2(v_0^2+w_0^2)(v^2+w^2)-4\gamma^2(v_0^2v^2+w_0^2w^2)\Big]\\
&+2(\gamma^4-1)\sinh 2(\chi_0-\chi)\cos 2(\theta_0-\theta)v_0w_0(w^2-v^2)\\
&-4\gamma(\gamma^2+1)\cosh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)v_0w_0(w^2+v^2)\Big\}\\
&+4ij(\gamma^2+1)\sinh 2(\chi_0-\chi)\Big[2v_0w_0\sin 2(\theta_0-\theta)\\
&-(\gamma^2-1)(w_0^2-v_0^2)\cos 2(\theta_0-\theta)\Big]\underbrace{vw}\\
&+8\gamma ij(\gamma^2+1)\cosh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)(v_0^2+w_0^2)\underbrace{vw};
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\psi\overline{\psi}(\psi_0\overline{\psi}+\overline{\psi}_0\psi)=2\left[(\gamma^2-1)(v^2+w^2)+2\gamma ij(w^2-v^2)\right]\cdot\\
&\times\Big\{(\gamma^2-1)\cosh(\chi_0-\chi)\cos(\theta_0-\theta)(w_0w+v_0v)\\
&-2\gamma\sinh(\chi_0-\chi)\sin(\theta_0-\theta)(w_0w-v_0v)\\
&-(\gamma^4+1)\cosh(\chi_0-\chi)\sin(\theta_0-\theta)(v_0w-w_0v)\\
&+ij\Big[(\gamma^2-1)\sinh(\chi_0-\chi)\sin(\theta_0-\theta)(w_0w+v_0v)\\
&+2\gamma\cosh(\chi_0-\chi)\cos(\theta_0-\theta)\cdot(w_0w-v_0v)\\
&+(\gamma^2+1)\sinh(\chi_0-\chi)\cos(\theta_0-\theta)(v_0w-w_0v)\Big]\Big\};
\end{aligned}
\tag{34}
$$

where the fields have the general form (5), with $\psi_0=\psi_0(\gamma,v_0,w_0;\theta_0,\chi_0)$, and $\psi=\psi(\gamma,v,w;\theta,\chi)$; note the presence of ij-hybrid terms in Eqs. (33), and (34). Furthermore, the presence of mixed terms vw (both real and hybrid) in the quadratic form (33) prevents us from determining the masses of v and w; note also that in relation to the same mixed terms, one must exploit the freedom of choosing the circular and hyperbolic parameters, and v.e.v. (v_0,w_0), in order to obtain the simultaneous vanishing of the real and hybrid terms of the form vw.

Finally the equations of motion are given by

$$
\left[\Box+\frac{\lambda}{6}\left(\psi\overline{\psi}-\frac{6am^2}{\lambda}\right)\right]\psi=0,\quad \Box=\partial^2t-\nabla^2,
\tag{35}
$$

with an energy-momentum tensor given by

$$
2T_{ij}=\partial_i\psi\cdot\partial_j\psi-g_{ij}\left[\frac{1}{2}\partial^\kappa\psi\cdot\partial_\kappa\psi-\frac{\lambda}{4!}\left(\psi\cdot\overline{\psi}-\frac{6am^2}{\lambda}\right)^2\right];
\tag{36}
$$

in particular, the energy density reads

$$
\mathcal{E}=2T_{00}=\frac{1}{2}|\partial_t\psi|^2+\frac{1}{2}\nabla\psi\cdot\nabla\overline{\psi}+\frac{\lambda}{4!}\left(\psi\overline{\psi}-\frac{6am^2}{\lambda}\right)^2.
\tag{37}
$$

Equations (35), (36), and (37) will allow us to study the possible formation of global strings with finite energy in Sect. 4.6.

4.1 The case $\gamma^2=1$, $m_R^2=0$, $\lambda_H=0$: hyperbolic version of the Mexican hat

The case $\gamma^2=1$ is special, since the norm (13) reduces to the hyperbolic part, a purely hybrid expression. In relation to Eq. (22) that defines the degenerate vacuum, the restriction $\gamma^2=1$ implies more restrictions on the right-hand side; a nontrivial choice is $m_R=0$, and $\lambda_H=0$; then the hybrid component of the potential V_H (Eq. (16)) vanishes, and its real part reduces to

$$
V=V_R=a\gamma m_H^2(v^2-w^2)-\frac{\lambda}{6}(v^2-w^2)^2,\quad V_H=0;
\tag{38}
$$

additionally the minima are defined by the hyperbola (23), irrespective of the signs of the parameters $(a,\gamma,\lambda_R=\lambda)$;

$$
a\gamma(v_0^2-w_0^2)=\frac{3}{\lambda}m_H^2;
\tag{39}
$$

the hyperbola with $a\gamma>0$ is the conjugate of that with $a\gamma<0$, and are related by a $\frac{\pi}{2}$-rotation in the plane $v-w$. In the appropriate field variables (Θ_1,Θ_2), the hyperbola takes the form $\Big(\frac{\Theta_1}{\sqrt{|\frac{6m_H^2}{\lambda}|}}\Big)^2-\Big(\frac{\Theta_2}{\sqrt{|\frac{6m_H^2}{\lambda}|}}\Big)^2=1$, and thus is rectangular with eccentricity $\sqrt{2}$, with the vertices localized at $\pm\sqrt{|\frac{6m_H^2}{\lambda}|}$, which coincide with the position of the two minima in the original Spontaneous Symmetry Breaking (SSB) scenario described in terms of the real field ϕ in Fig. 1. From the substitution of the expression (39) into Eq. (38), one obtains the energy of vacuum,

$$
V(v_0,w_0)=\frac{3}{2}\frac{m_H^4}{\lambda}.
\tag{40}
$$

For $\lambda<0$, the hyperbola corresponds to the global minima for the energy, and its value corresponds to the value for the two lowest energy states in the original SSB scenario described in Fig. 1. For the stationary points described in Eq. (19) we have $\det\mathcal{H}_{(V_R)}(0,0)=-4m_H^4$, and thus the zero-energy point is a saddle point.

In Fig. 2, the potential is not the Mexican hat potential, particularly in relation to the existence of two connected regions for the possible vacuum states, and that each region is noncompact, as opposed to the compact circle associated to the

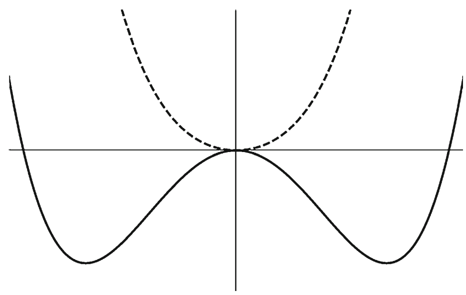

Fig. 1 The *dashed* potential shows the profile for the cases $m^2 \geqslant 0$, and the *continuous* potential shows the case $m^2 < 0$; in the first case the vacuum manifold corresponds to a single point, thus $\pi_0 = 1, \pi_{n\geqslant 1} = \{0\}$, and in the second case, it corresponds to two points (the 0-sphere), with $\pi_0 = 2, \pi_{n\geqslant 1} = \{0\}$

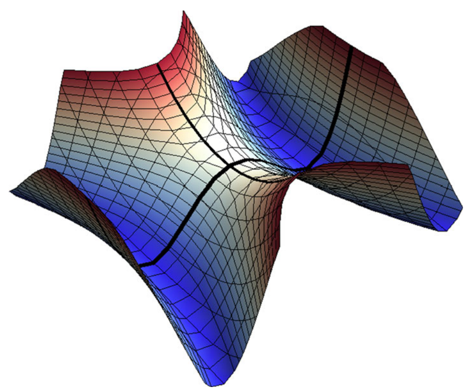

Fig. 2 The potential (38) for $\lambda < 0$; the stable states are localized at the blue region, a hyperbola described by Eq. (39)

conventional $U(1)$-symmetry. Let us see how the two scenarios described in the conventional Lagrangian (12) are contained in certain sense in this hypercomplex re-formulation; if the original Lagrangian corresponds to an Unbroken Exact Symmetry (UES) scenario with $L(\phi, a = 1)$, then the potential $V(\phi, a = 1)$ has a minimum (see Fig. 1), and it is shown as a *bold-face* curve embedded in Fig. 2; the hypercomplex extension of ϕ leads to a new potential, transforming the original stable minimum into a saddle point as the zero-energy point, and with the appearance of other stable minima at the blue region. Similarly, the usual SSB scenario with $V(\phi, a = -1)$ in Fig. 1 with two minima is shown also as a *bold-face* curve in Fig. 2. Thus, the two minima have been extended to an infinite number of minima in two disconnected regions colored in blue, which correspond to the hyperbola in Eq. (39); the original maximum point of the potential $V(\phi, a = -1)$ is the saddle point for the new potential. Hence, the Lagrangian (14) is build originally on a saddle point (the crossing point of the *bold-face* curves), and the choice of a stable vacuum in the blue region is mandatory, and we proceed now to the study of the realizations of the symmetry breakdown.

In this case Eqs. (33), and (34) reduce to

$$\begin{aligned}
&\overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2 = 4\cosh 2(\chi_0-\chi)\cos 2(\theta_0-\theta)\\
&\quad\times\left[\frac{3a\gamma}{\lambda}m_H^2(v^2-w^2) - (v_0^2+3w_0^2)w^2 - (3v_0^2+w_0^2)v^2\right]\\
&\quad -16\gamma\sinh 2(\chi_0-\chi)\cdot\cos 2(\theta_0-\theta)v_0w_0(v^2+w^2)\\
&\quad +16\cos 2(\theta_0-\theta)\\
&\quad\times\left[2v_0w_0\cosh 2(\chi_0-\chi)+\gamma(v_0^2+w_0^2)\sinh 2(\chi_0-\chi)\right]\underbrace{vw}\\
&\quad +4ij\sinh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)\\
&\quad\times\left[\frac{3a\gamma}{\lambda}m_H^2(v^2-w^2)-(v_0^2+w_0^2)(v^2+w^2)-2(v_0^2v^2+w_0^2w^2)\right]\\
&\quad +16\gamma ij\cosh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)\\
&\quad\times\left[(v_0^2+w_0^2)vw - v_0w_0(v^2+w^2)\right]\\
&\quad +16ij\sinh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)v_0w_0\underbrace{vw},
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\psi\overline{\psi}(\psi_0\overline{\psi}+\overline{\psi}_0\psi) = 8\gamma ij(w^2-v^2)\\
&\quad\times\Big[\gamma\sinh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)(v_0v-w_0w)\\
&\quad +\cosh 2(\chi_0-\chi)\sin 2(\theta_0-\theta)(w_0v-v_0w)\\
&\quad +ij\sinh 2(\chi_0-\chi)\cos 2(\theta_0-\theta)(v_0w-w_0v)\\
&\quad +\gamma ij\cosh 2(\chi_0-\chi)\cos 2(\theta_0-\theta)(w_0w-v_0v)\Big];
\end{aligned}\tag{42}$$

We can see that only some mixed terms vw (both real and hybrid), can be gauged away by fixing the hyperbolic parameters $\chi = \chi_0$; a particular choice, say $\chi_0 = 0$, leads to a break down of the symmetry $SO^+(1, 1)$, remembering that in the case $\gamma^2 = 1$, $U(1)$ corresponds to a spurious rotation. Even so the remanent $U(1)$ parameters must be fixed in Eq. (41) by demanding the vanishing of the remaining mixed term $v \cdot w$, for example, by fixing

$$\begin{aligned}
&\theta = \theta_0 = 0,\quad v_0 = 0,\quad w_0^2 = -\frac{3a\gamma}{\lambda}m_H^2;\\
&\to \frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2+\psi_0^2\overline{\psi}^2) = a\gamma m_H^2(w^2+v^2),
\end{aligned}\tag{43}$$

with $a\gamma = 1$, and $\lambda < 0$; hence, taking into account that $-\frac{am^2}{2}\psi\overline{\psi} = a\gamma m_H^2(w^2 - v^2)$, the mass matrix determined by Eq. (32) for the Lagrangian (14) reads

$$\begin{pmatrix} 0 & 0 \\ 0 & -2m_H^2w^2 \end{pmatrix},\tag{44}$$

which corresponds to a massive ordinary scalar field w, and a massless field v. Conversely, if the field v develops a non-zero vacuum expectation value with $v_0^2 = \frac{3a\gamma}{\lambda}m_H^2$, $a\gamma = -1$, and $w_0 = 0$, then we shall have an ordinary massive term for the field v, and the field w will be now massless; in this case the mass matrix reads

$$\begin{pmatrix} -2m_H^2v^2 & 0 \\ 0 & 0 \end{pmatrix}.\tag{45}$$

If the spurious $U(1)$ rotations are broken for example with the choice (43), then the mixed terms vw in the real part of Eq. (33) vanish trivially, and the complete mass term in Eq.

(32) reduces to

$$-\frac{am^2}{2}\psi\overline{\psi}+\frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2+\psi_0^2\overline{\psi}^2)$$
$$=m_H^2\Big\{[\cosh 2(\chi-\chi_0)-1]v^2+[1+\cosh 2(\chi-\chi_0)]w^2$$
$$-4\gamma\sinh 2(\chi-\chi_0)vw\Big\}, \tag{46}$$

which is fully real, without ij-hybrid terms; similarly the cubic expression (42) is fully real under the choice (43). Therefore, in relation to the expression (46), whereas the hyperbolic rotation symmetry is not conveniently fixed, the masses of the fields v and w cannot be determined due to the presence of the mixed term vw. However, the choice $\chi_0 = 0$ fixes a point on the hyperbola, and the condition $\chi = 0$ leads to the same mass matrix described previously in Eq. (44), and similarly for the mass matrix in Eq. (45).

In the spontaneous symmetry breaking scenarios described above, the hyperbolic parameters are always chosen with finite values, in similarity with the compact circular parameters; however, the hyperbolic parameters take in principle values on a non-compact interval, with $|\chi| < \infty$, and $|\chi_0| < \infty$. Hence, the remanent $SO^+(1,1)$ symmetry in Eq. (46) can be spontaneously broken by considering that $\chi_0 \to +\infty$, and hence $\sinh\chi_0 \approx \cosh\chi_0 \approx \frac{e^{\chi_0}}{2}$, and for the hyperbolic functions in Eq. (46) we have

$$\cosh 2(\chi-\chi_0)\approx\frac{e^{2\chi_0}}{2}[\sinh\chi-\cosh\chi]^2,$$
$$\sinh 2(\chi-\chi_0)\approx-\frac{e^{2\chi_0}}{2}[\sinh\chi-\cosh\chi]^2, \tag{47}$$

which are divergent, unless $\chi \to +\infty$, and hence $\sinh\chi - \cosh\chi \approx 0$; thus $\lim[\cosh 2(\chi-\chi_0)] = 1$, and $\lim[\sinh 2(\chi-\chi_0)] = 0$; therefore the mass matrix obtained from (46) coincides with that obtained previously with the choice $\chi = \chi_0 = 0$. Similarly, in the case of the other asymptotic limit $\chi_0 \to -\infty$ we have $-\sinh\chi_0 \approx \cosh\chi_0 \approx \frac{e^{-\chi_0}}{2}$, and the hyperbolic expressions depend now on $\sinh\chi + \cosh\chi$, which goes to zero in the limit $\chi \to -\infty$.

The hyperbola has two connected regions, and hence $\pi_0 = 2$; each connected region is topologically equivalent to R, which has trivial homotopy groups, thus $\pi_n = 0$, for $n \geq 1$; there are no topological defects associated with continuous symmetries, and only the *domains walls* are possible due to the nontriviality of π_0.

Since the structure of this theory is basically hyperbolic, it reduces in essence to the substitution of the ordinary imaginary unit i, by the new hyperbolic unit j; as already mentioned, this simple substitution generates nontrivial results in diverse scenarios [1,2,7–9]. In the hyperbolic "$\lambda\phi^4$" theory at hand, the topological defects such as strings, cannot form; these defects are unavoidable in the breaking of an Abelian $U(1)$ symmetry. Hence, a phenomenological field theory based on a hyperbolic symmetry may be a way of evading the problem of the insignificant empirical evidence of cosmic strings, contrary to the predictions of many field theory models. The possible cosmological implications of these speculations are discussed in the concluding remarks.

Now we are going beyond the substitution of i by j, and the incorporation of j described previously in terms of a commutative ring will allow us to study the hyperbolic deformation of the Mexican hat potential; the parameter γ will govern the competition between the circular and hyperbolic contributions.

4.2 The case $\gamma^2 \neq 1$, $(v_0 \neq 0, w_0 = 0)$, and $\lambda_R = \lambda_H$: hyperbolic deformation of the Mexican hat

The restrictions $\gamma^2 \neq 1$ and $\gamma \neq 0$ imply that the norm (13) will have nontrivial contributions from the *circular* and *hyperbolic* parts; the additional restriction $\lambda_H = \lambda_R \equiv \lambda$ will allow us to simplify the analysis; hence Eqs. (28) and (29) for the ratio between the squared masses and squared expectation value, reduce to

$$\frac{m_H^2}{m_R^2}=\frac{\gamma^2-2\gamma-1}{\gamma^2+2\gamma-1},\quad v_0^2=\frac{6m_R^2}{a\lambda(1-\gamma^2-2\gamma)}; \tag{48}$$

the first equation above defines a positive quotient, restricting the values of γ on the right-hand side; similarly positivity on the left-hand side of the second equation implies the inequality

$$a\lambda(1-\gamma^2-2\gamma)>0. \tag{49}$$

Other relevant quantities for inducing a stable vacuum are $\det\mathcal{H}$ for the potentials at the zero-energy point,

$$\det\mathcal{H}_{v_R}(v_0=0,w_0=0)=\frac{m_R^4(\gamma^2+1)^2}{(1-\gamma^2-2\gamma)^2}$$
$$\times[\gamma^4+4\gamma^3-6\gamma^2-4\gamma+1], \tag{50}$$
$$\det\mathcal{H}_{v_H}(v_0=0,w_0=0)=\frac{m_R^4(\gamma^2+1)^2}{(1-\gamma^2-2\gamma)^2}$$
$$\times[\gamma^4-4\gamma^3-6\gamma^2+4\gamma+1]; \tag{51}$$

if $\det\mathcal{H} > 0$, the zero-energy point will be a minimum or maximum; if $\det\mathcal{H} < 0$ it will be a saddle point. The polynomials of γ in Eqs. (48), and (49), and those that determine the signs of $\det\mathcal{H}$ in Eqs. (50) and (51), are shown in Fig. 3.

In Fig. 3, the vertical blue asymptote represents one root of the polynomial $(\gamma^2+2\gamma-1)$, $\gamma = \sqrt{2}-1$; this singular point is out of the interval of interest. The arrow on the left-hand side points out a root of $\det\mathcal{H}_{V_H}(0,0)$, $\gamma_H = 1+\sqrt{2}-\sqrt{2(2+\sqrt{2})} \approx -0.1989$; similarly on the right-hand side an arrow points out a root of $\det\mathcal{H}_{V_R}(0,0)$ localized at $\gamma_R = -\gamma_H \approx 0.1989$; within this symmetric interval all

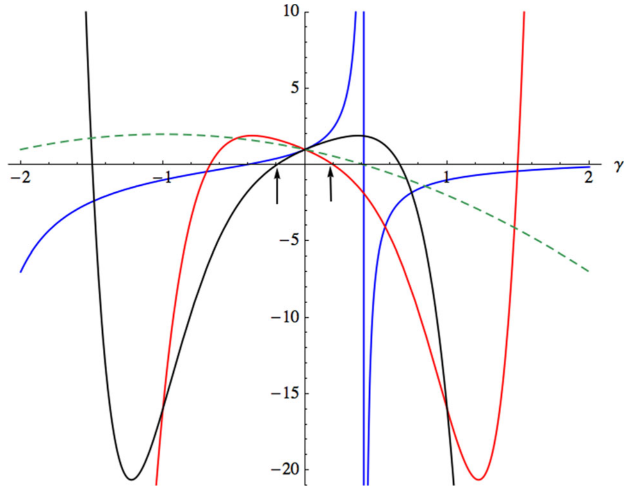

Fig. 3 The *continuous red curve* represents essentially det $\mathcal{H}_{V_R}(0,0)$; the *continuous black curve* represents det $\mathcal{H}_{V_H}(0,0)$. The *continuous blue curve* represents the polynomial $(\gamma^2-2\gamma-1)/(\gamma^2+2\gamma^2-1)$, the ratio in Eq. (48); the *dashed curve* represents the polynomial $(1-\gamma^2-2\gamma)$, Eq. (49); the *arrows* point out the interval where all polynomials are positive

polynomials are positive, and a stable vacuum will be induced for the potentials. In this interval, the inequality (49) implies that

$$a\lambda > 0. \tag{52}$$

Under these simplifications the potentials will take the following form:

$$V_R = \frac{a}{2}P_R^v m_R^2 v^2 + \frac{a}{2}P_R^w m_R^2 w^2 + \frac{\lambda}{6}\Bigg[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2 - \gamma(\gamma^2-1)(w^4-v^4)\Bigg]; \tag{53}$$

$$V_H = \frac{a}{2}P_H^v m_R^2 v^2 + \frac{a}{2}P_H^w m_R^2 w^2 + \frac{\lambda}{6}\Bigg[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2 + \gamma(\gamma^2-1)(w^4-v^4)\Bigg]; \tag{54}$$

where the polynomials P are defined as

$$P_R^v = \frac{\gamma^4+4\gamma^3-6\gamma^2-4\gamma+1}{\gamma^2+2\gamma-1}, \quad P_R^w = \frac{(\gamma^2+1)^2}{\gamma^2+2\gamma-1};$$
$$P_H^v = \frac{\gamma^4-4\gamma^3-6\gamma^2+4\gamma+1}{\gamma^2+2\gamma-1}, \quad P_H^w = P_R^w; \tag{55}$$

the potentials (53), and (54) are shown in Fig. 4 as functions of (v, w), for a value of γ in the interval $(\gamma_H, -\gamma_H)$; similarly the polynomials (55) are shown in Fig. 5. The values $\pm\gamma_H$ are critical, since Eqs. (50), and (51) vanish, and thus the character of a local maximum for the zero-energy point and the form of the potentials with stable minima shown in Fig. 4 are not guaranteed; therefore one must be careful on taking the limit $\gamma \to \pm\gamma_H$.

Furthermore, the vacuum energies are given by

$$V_R(\pm v_0, 0) = \frac{-3m_R^4}{2\lambda}\frac{P_R^v(\gamma)}{\gamma^2+2\gamma-1},$$
$$V_H(\pm v_0, 0) = \frac{-3m_R^4}{2\lambda}\frac{2(\gamma^2-1)^2}{(\gamma^2+2\gamma-1)^2}; \tag{56}$$

hence, the depth of the red regions in Fig. 4 depends on γ; the polynomials that deform the conventional vacuum energies in the above expressions are shown in Fig. 6. For $\gamma = 0$ one can recover from V_R the vacuum energy for the usual $U(1)$ field theory, since $\left.\frac{P_R^v(\gamma)}{(\gamma^2+2\gamma-1)}\right|_{\gamma=0} = 1$. In the subinterval $[\gamma_H, 0]$ the deformation is small with respect to the deformation in

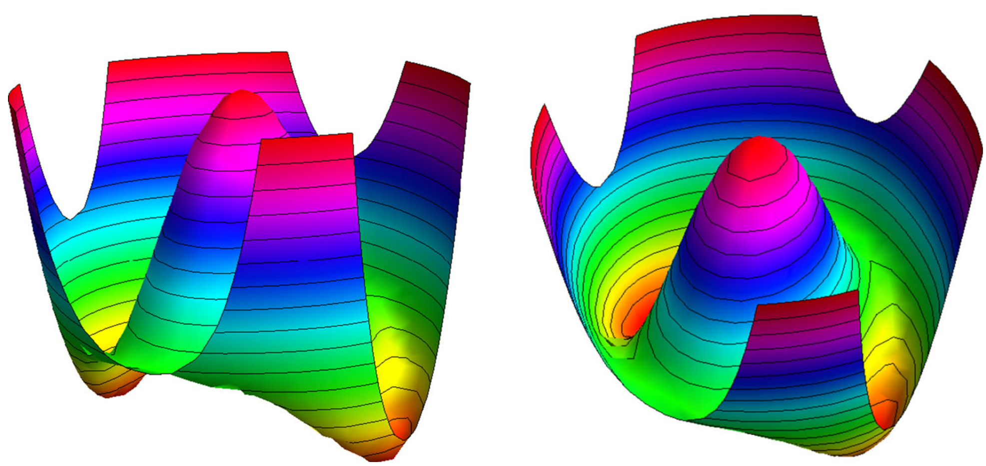

Fig. 4 The potentials V_R/m_R^2, and V_H/m_R^2 in the interval $(\gamma_H, -\gamma_H)$ have essentially the same qualitative aspect, which we show here from two different perspectives: the zero-energy point corresponds to the central peak of the potential; the two minima are localized at the *bottom* in the two red regions: $(\pm v_0, 0)$. The inequality (52) is satisfied with $a = 1$ and $\lambda > 0$; the choice $a = -1$ and $\lambda < 0$ turns the potentials upside down

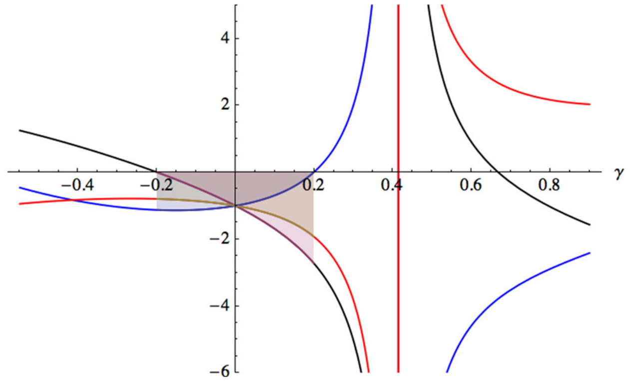

Fig. 5 The mass polynomial coefficients: the *blue curve* represents P_R^v; the *black curve* represents P_H^v, and the *red curve* represents P_R^w. In the *shadowed* interval $(\gamma_H, -\gamma_H)$ all polynomials are negative, and $P_R^v(-\gamma_H) = 0 = P_H^v(\gamma_H)$, $P_R^v(\gamma_H) \approx -1.1252$,$P_H^v(-\gamma_H) \approx -2.7165$, and all polynomials satisfy $P_R^v(0) = P_H^v(0) = P_R^w(0) = -1$

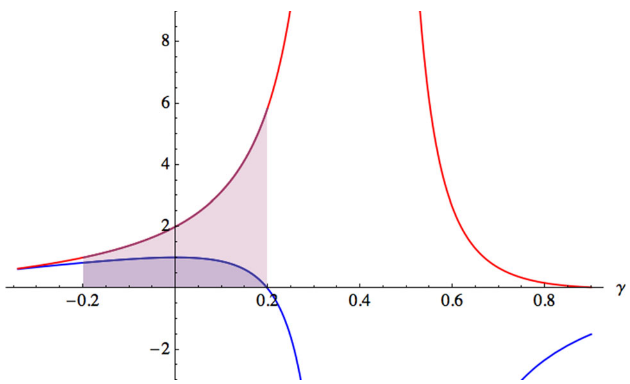

Fig. 6 The *blue curve* represents $P_R^v(\gamma)/(\gamma^2+2\gamma-1)$; the *red curve* represents $\frac{2(\gamma^2-1)^2}{(\gamma^2+2\gamma-1)^2}$. In the interval $(\gamma_H, -\gamma_H)$ the polynomials take positive values, and the vacuum energies are finite, even at the limits $\pm\gamma_H$

the subinterval $[0, -\gamma_H]$; the polynomial is varying continuously between the values [.828, 1] in the first subinterval, and between the values [1, 0] in the second subinterval, since $P_R^v(-\gamma_H) = 0$. Furthermore, from Fig. 6 it is evident that the deformation of V_H is strictly bigger than that of V_R; thus, the depth of the red regions in Fig. 4 is higher for V_H. Such a difference in the deformation is maximum for $-\gamma_H$, and minimum for γ_H.

The circular and hyperbolic rotations can be spontaneously broken by the choice

$$\sin 2(\theta - \theta_0) = 0; \quad \sinh 2(\chi - \chi_0) = 0; \tag{57}$$

in addition to the vacuum expectation values ($v_0 \neq 0$, $w_0 = 0$); thus all mixed terms vw (both real and hybrid) can be gauged away, reducing the quadratic terms in Eq. (32) to the canonical form,

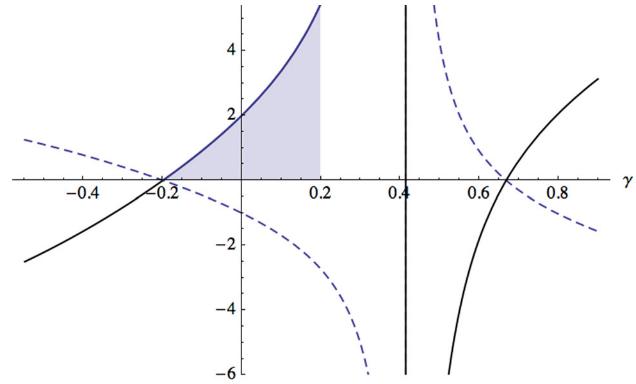

Fig. 7 The *continuous curve* represents the mass polynomial coefficient $-2P_H^v(\gamma)$ after SSB: the *dashed curve* represents the mass polynomial coefficient $P_H^v(\gamma)$ before the SSB; the case $\gamma = 0$ reproduces the usual SSB of $U(1)$. The field w is a fully massless field for any γ in the interval

$$-\frac{am^2}{2}\psi\overline{\psi} + \frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2) = a\left(\frac{1-\gamma^2}{2}m_R^2 - \gamma m_H^2\right)v^2$$
$$+a\left(\frac{1-\gamma^2}{2}m_R^2 + \gamma m_H^2\right)w^2 + ija\left[\left(\frac{1-\gamma^2}{2}m_H^2 + \gamma m_R^2\right)v^2\right.$$
$$\left.+\left(\frac{1-\gamma^2}{2}m_H^2 - \gamma m_R^2\right)w^2\right] = ij\frac{a}{2}(-2P_H^v m_R^2)v^2; \tag{58}$$

where the last equality follows from the mass relation (48); this expression must be compared to the mass terms in Eq. (53), and (54). Therefore, the field w is massless in both senses, real and hybrid. The field v has duplicated its hybrid mass with a change of sign; note that the real mass of v has disappeared. The duplication of the mass with a change of sign for any γ in the allowed interval is shown in Fig. 7. The mass that arises from SSB in Eq. (58) is actually a *running* mass, since the polynomial P_H^v takes values in the interval $(0, -2.7165)$; hence the mass is running in the interval $(0, 2.7165m_R^2)$, from a nearly massless field to a “heavy” field. Strict masslessness is prohibited, since the value $-\gamma_H$ is critical in relation to the form of the potentials with local stable minima required for the spontaneous symmetry breaking.

From Eq. (13) we can realize that in the limit $\gamma \to 0$, the usual norm of an ordinary $U(1)$ complex field can be recuperated; in this sense the case with $\gamma \neq 0$ can be understood as a hyperbolic deformation around the usual formulation. Furthermore, according to Eq. (48), in such a limit there is no difference between m_R and m_H; the vaccum expectation value v_0^2 will reduce to the usual $U(1)$ expression. Similarly the inequality (49) will reduce to the inequality (52), which corresponds to the usual constraint in the $U(1)$ field theory. Likewise, all polynomials (55) reduce to -1, and consequently there is no difference between V_R and V_H. Therefore, the usual Mexican hat potential can be recuperated from the

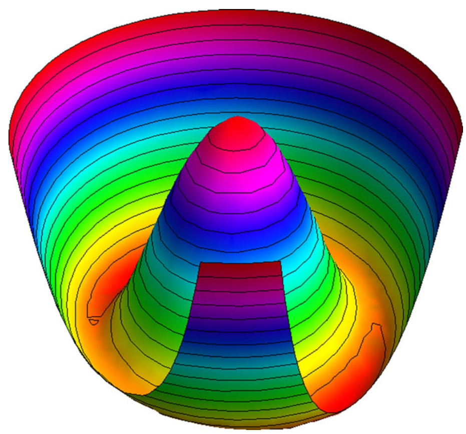

Fig. 8 The mexican hat potential as the limit of V_R and V_H as $\gamma \to 0$

deformed version described in Fig. 4; this process is shown in Fig. 8.

4.3 The case $\gamma^2 \neq 1$, ($v_0 = 0$, $w_0 \neq 0$), and $\lambda_R = \lambda_H$

Along the same lines, the use of Eqs. (30), and (31) lead essentially to the same expressions (48), (49), (50), and (51), with the change $\gamma \to -\gamma$; Fig. 3 is essentially the same; for example the red and black curves will convert to each other, without changing the interval $(\gamma_H, -\gamma_H)$; within this interval all new polynomials are negative. Furthermore, the inequality (52) remains valid, and the potentials can be described in terms of the same mass polynomials appearing in Eqs. (53) and (54);

$$V_R = \frac{a}{2} Q^v_R m^2_R v^2 + \frac{a}{2} Q^w_R m^2_R w^2 + \frac{\lambda}{6}\Big[\cdots\Big]; \tag{59}$$

$$V_H = \frac{a}{2} Q^v_H m^2_R v^2 + \frac{a}{2} Q^w_H m^2_R w^2 + \frac{\lambda}{6}\Big[\cdots\Big]; \tag{60}$$

where the dots represent exactly the same expressions for the λ-terms in Eqs. (53), and (54); the polynomials Q are defined as

$$Q^v_R = P^w_R(\gamma \to -\gamma), \quad Q^w_R = P^v_R(\gamma \to -\gamma);$$
$$Q^v_H = Q^v_R, \qquad Q^w_H = P^v_H(\gamma \to -\gamma); \tag{61}$$

these potentials have the same form shown in Fig. 4. However, in spite of the similarities, there will be an important difference; although the field v will be massless as expected, the field w will develop a mass with both parts, real and hybrid; explicitly we have

$$-\frac{am^2}{2}\psi\overline{\psi} + \frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2)$$
$$= \frac{a}{2} m^2_R(-2Q^w_R)w^2 + \frac{a}{2} ijm^2_R(-2Q^w_H)w^2; \tag{62}$$

which are shown in Fig. 9; the real mass in Eq. (62) is running in the interval $(0, 1.1252 m^2_R)$, and the hybrid mass is running in the interval $(0, 2.7165 m^2_R)$. Note that when the real mass goes to zero as $\gamma \to \gamma_H$, the hybrid mass goes to its maximum value, and coversely in the limit $\gamma \to -\gamma_H$; the masses coincide in the value m^2_R for $\gamma = 0$.

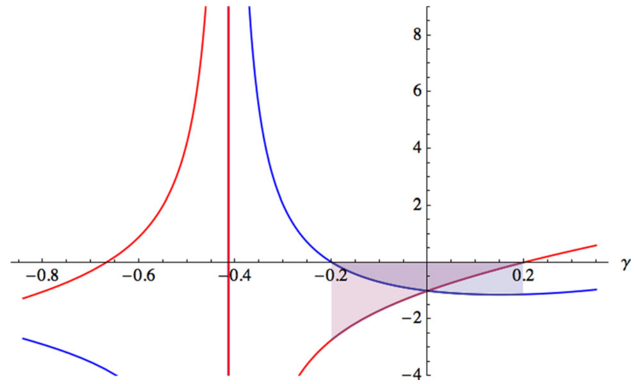

Fig. 9 Q^w_R corresponds to the *blue curve*, with values in the interval $(0, -1.1252)$; Q^w_H is represented in *red*, with values in the interval $(0, -2.7165)$

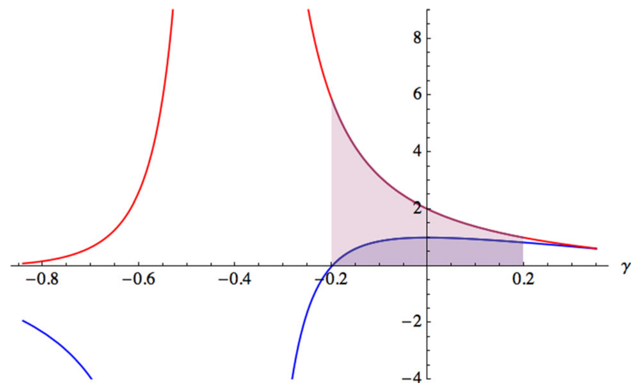

Fig. 10 The description of the vacuum energies given in Fig. 6 is essentially valid for this case, in relation to the behavior in the subintervals $[\gamma_H, 0]$, and $[0, -\gamma_H]$, and the differences between the values of V_R and V_H in the vacuum

In this case the vacuum energies are given by

$$V_R(0, \pm w_0) = \frac{-3m^4_R}{2\lambda}\frac{P^v_R(-\gamma)}{\gamma^2 - 2\gamma - 1},$$
$$V_H(0, \pm w_0) = \frac{-3m^4_R}{2\lambda}\frac{2(\gamma^2-1)^2}{(\gamma^2 - 2\gamma - 1)^2}; \tag{63}$$

which can be obtained directly from Eq. (56) through the change $\gamma \to -\gamma$; the polynomials in Eq. (63) are shown in Fig. 10.

4.4 The vacuum manifold for $\gamma \in (-\gamma_H, \gamma_H)$

In the usual formulation of the theory with a conventional complex field, the degenerate vacuum is identified with the bottom of the Mexican hat potential, the S^1 compact potential defined by the constraint $|v_0 + iw_0|^2 = \frac{6m^2}{\lambda}$, and parametrized by $v_0 = \sqrt{\frac{6m^2}{\lambda}} \cos\theta$, and $w_0 = \sqrt{\frac{6m^2}{\lambda}} \sin\theta$; this case corresponds essentially to the choice $\gamma = 0$ in the formulation at hand. However, in the present case the hypercomplex field is undetermined additionally by a hyperbolic phase, leading to a two-dimensional manifold for the degenerate vacuum; this case corresponds to the choice $\gamma \neq 0$ in Eq. (4). We shall see that the vacuum manifold will correspond to a non-compact space containing the hyperbola and the circle as factor spaces, and embedded in a four-dimensional ambient space.

A parametrization for the vacuum manifold can be given by the expression (5), with $\gamma \in (-\gamma_H, \gamma_H)$, and with $\lambda\psi_0\overline{\psi}_0 = -6am^2$; one can consider additionally the choice (29) with $w_0 = 0$, and $v_0 \neq 0$;

$$\begin{aligned}\psi_0 = v_0 \Big\{ &(\cos\theta_0 \sinh\chi_0 - \gamma \sin\theta_0 \cosh\chi_0) \\ &+ i(\gamma \cos\theta_0 \cosh\chi_0 + \sin\theta_0 \sinh\chi_0) \\ &+ j(\cos\theta_0 \cosh\chi_0 - \gamma \sin\theta_0 \sinh\chi_0) \\ &+ ij(\sin\theta_0 \cosh\chi_0 + \gamma \cos\theta_0 \sinh\chi_0) \Big\}. \end{aligned} \tag{64}$$

One can embed the vacuum manifold as a two-dimensional subspace of the real four-dimensional space defined by the four components of the field $\psi_0 = x_0 + iy_0 + jz_0 + ijw_0 \rightarrow (x_0, y_0, z_0, w_0)$, and the result must be projected from 4D to 3D in order to be visualized, and to gain insight about its geometrical and topological properties. The more practical and direct method that can be used is the projection of the 2-manifold into the three-dimensional hyperplanes that define the four coordinate hyperplanes in the four-dimensional ambient space, through $(x_0, y_0, z_0, w_0) \rightarrow (x_0, y_0, z_0)$, and similarly into the other three hyperplanes. Although in general such projections can be different depending of the *orientation* of the 2-manifold with respect to the coordinate hyperplanes, in this case, the four projections coincide to each other, and are represented in Fig. 11. This projection is a sort of *product* of a hyperbola and a circle; it can be visualized also as two-dimensional planes embedded in three dimensions and sharing a hole. The *self-intersection* is an effect of the projection into a three-dimensional hyperplane; in the original four-dimensional ambient space the two-manifold has no such self-intersections. This case is similar to the very known Klein bottle, which cannot be realized in R^3 without intersecting itself.

We remark that the constraint defining the degenerate vacuum $\lambda\psi_0\overline{\psi}_0 = -6am^2$, retains the full symmetry $SO^+(1,1) \times U(1)$, and the vacuum manifold is transformed into itself by the action of these transformations, and we have an infinite number of possible vacuum states with the same energy. The vacuum manifold is homotopic to S^1, and the *string defects* can be formed as topological defects; we shall analyze in detail this topic in Sects. 4.6 and 5.4.

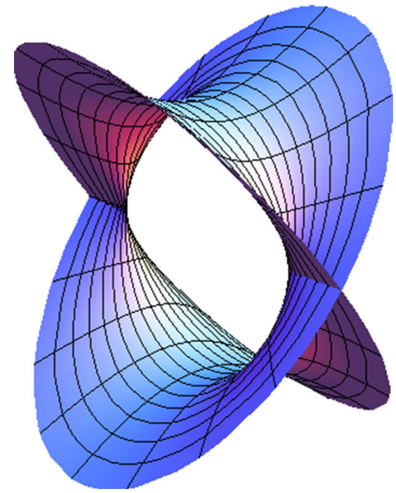

Fig. 11 The degenerate vacuum as a non-compact and not simply connected two-manifold embedded in the 3d space. Usually only the compact transversal sections related with $U(1)$ have been considered as the vacuum; for this manifold of genus 1, we have $\pi_0 = 1, \pi_1 = \mathcal{Z}$, and $\pi_{n\geqslant 2} = 0$, since it is homotopic to S^1

4.5 Polar parametrization for the fields

In the previous cases the hypercomplex field ψ is described in terms of Cartesian components (v, w) as real variable fields; now we use a *polar* decomposition, considering that we have at hand only two real variable fields. Thus, following the ideas at the end of Sect. 3, for an expression of the form $(\rho_R + ij\rho_H)e^{i\xi}e^{j\eta}$, where ρ_R, ρ_H, ξ, and η, are real field variables, one must choose two of them as constants; similarly, the polar form for the vacuum field reads $\varphi_0 = (\rho_0^R + ij\rho_0^H)e^{i\xi_0}e^{j\eta_0}$.

Case $\gamma^2 = 1$: For this case studied previously in the Sect. 4.1, we see that the norm of the dynamical field reduces to $\psi\overline{\psi} = 2ij\gamma(w^2 - v^2)$; additionally the norm of the corresponding polar form will reduce to

$$\psi\overline{\psi} = (\rho_R^2 - \rho_H^2) + 2ij\rho_R\rho_H; \tag{65}$$

thus we have $\rho_R = \gamma\rho_H$, and consequently

$$\psi = (\gamma + ij)\rho e^{i\xi}e^{j\eta}, \qquad \rho \equiv \rho_H = \sqrt{w^2 - v^2}; \tag{66}$$

therefore, in this case only one degree of freedom is encoded in the polar part, and the remaining degree of freedom will be encoded in one of the phases; similarly the vacuum field takes the form $\psi_0 = (\gamma + ij)\rho_0 e^{i\xi_0}e^{j\eta_0}$, with the constraint $\rho_0^2 = \frac{6m_H^2}{\lambda}$.

We consider first the case $\psi(x) = (\gamma + ij)\rho(x)e^{i\xi(x)/\rho_0}e^{j\eta}$, with η constant, and excitations about the ground state with vanishing phases, $\xi_0 = 0 = \eta_0$, and hence the circular and hyperbolic rotations are broken. Now, we write the dynamical field as $\big(\gamma + ij\big)\big(\rho(x) + \rho_0\big)e^{i\xi(x)}e^{j\eta}$, and the Lagrangian becomes

$$\mathcal{L} = 2\gamma ij \int \mathrm{d}x^d \left[\frac{1}{2}\partial^i \rho \partial_i \rho + \frac{1}{2}\partial^i \xi \partial_i \xi - m_H^2 \rho^2 + \text{higher terms}\right], \quad (67)$$

and, as expected, we have massive radial oscillations ρ and circular *angle* massless oscillations ξ. There are no hyperbolic angle oscillations, due to the fixing $\eta = constant$, since they have disappeared completely from the Lagrangian. Similarly, if we fix $\xi = constant$, and $\eta \to \eta(x)/\rho_0$, then the oscillations around the same ground state are described by a Lagrangian of the form $L = \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\eta)^2 - m_H^2\rho^2$, plus higher terms, where the massive radial oscillations remain unchanged, but the hyperbolic angle oscillations appear now as massless modes with a global change of sign in the kinetic term.

Now one can enforce the restriction $\rho = constant$, and thus the two degrees of freedom will be encoded in the phases, and all fluctuations will lie in the *valley* directions; such directions are defined by the circular and hyperbolic lines on the vacuum manifold, and the fluctuations will correspond to purely massless modes, as expected. In this manner, if the two dynamical degrees of freedom are encoded in the phases, then

$$\begin{aligned}\psi &= (\gamma + ij)\rho_0 e^{i\xi(x)/\rho_0} e^{j\eta(x)/\rho_0} \\ &= (\gamma + ij)\rho_0 \left(1 + i\frac{\xi(x)}{\rho_0} + \cdots\right)\left(1 + j\frac{\eta(x)}{\rho_0} + \cdots\right), \\ &= (\gamma + ij)[\rho_0 + i\xi(x) + j\eta(x) + \cdots] \end{aligned} \quad (68)$$

where ξ, η, as well as their derivatives, are considered to be small real fields, and the dots represent higher order terms in the perturbations; therefore, the substitution into the Lagrangian yields the expression,

$$\mathcal{L}(\xi, \eta) = \gamma ij \int \mathrm{d}x^d \times \left[\partial^i \xi \partial_i \xi - \partial^i \eta \partial_i \eta - 2ij\partial^i \xi \partial_i \eta + \text{higher terms}\right], \quad (69)$$

since the mass terms have disappeared completely, the fields excitations in the *valley* directions are massless modes. Note, however, that the kinetic terms have a nonconventional form, due to the presence of a hybrid term that mixes the gradients of the field excitations (ξ, η); however, one can interchange the canonical form of quadratic gradients and the mixed form through the invertible mapping

$$\begin{pmatrix}\xi \\ \eta\end{pmatrix} \leftrightarrow \frac{1}{2}\begin{pmatrix}1 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}, \quad \partial^i \xi \partial_i \eta \leftrightarrow \partial^i \eta \partial_i \eta - \partial^i \xi \partial_i \xi. \quad (70)$$

Alternatively, the kinetic terms in the Lagrangian (69) can be rewritten in terms of the derivatives of the full phase $i\xi + j\eta$, by considering the identity $[\partial^k(i\xi + j\eta)][\partial_k(i\xi + j\eta)] = -[\partial^k(i\xi + j\eta)]\overline{[\partial_k(i\xi + j\eta)]} = -[\partial^i \xi \partial_i \xi - \partial^i \eta \partial_i \eta - 2ij\partial^i \xi \partial_i \eta]$, which corresponds essentially to the kinetic terms in Eq. (70). Furthermore, since we are considering small oscillations around a point of the vacuum manifold, the dynamics of the circular and hyperbolic oscillations are locally indistinguishable; in fact the expressions (68), (69), and (70) are invariant under the interchange $i\xi \leftrightarrow j\eta$. As an option, one can distribute the global hybrid term ij out of the integration in the Lagrangian (69), and hence the real and hybrid terms under integration will interchange their roles, without changing the physical conclusions.

Case $\gamma^2 \neq 1$: In this case the comparison between the norm $\psi\overline{\psi} = (\gamma^2 - 1)(v^2 + w^2) + 2ij\gamma(w^2 - v^2)$, and Eq. (65) leads to

$$\rho_R^2 - \rho_H^2 = (\gamma^2 - 1)(v^2 + w^2), \quad \rho_R \rho_H = \gamma(w^2 - v^2); \quad (71)$$

and similarly for the vacuum fields; these maps allow us to describe the fields with the pair (ρ_R, ρ_H) instead of the pair (v, w); now, we rewrite the dynamical field as $[\rho_R + \rho_0^R + ij(\rho_H + \rho_0^H)]$, leading to a Lagrangian of the form

$$\mathcal{L}(\rho_R, \rho_H) = \frac{1}{2}\int \mathrm{d}x^d \left[\partial^i \rho_R \partial_i \rho_R - \partial^i \rho_H \partial_i \rho_H + 2ij\partial^i \rho_R \partial_i \rho_H + 2m^2(\rho_R^2 - \rho_H^2 + 2ij\rho_R\rho_H) + \text{higher terms}\right]; \quad (72)$$

with $m^2 = m_R^2 + ijm_H^2$; hence, the two radial modes are massive as expected. This expression can be rewritten in a compact form in terms of the Hermitian field $\Pi \equiv \rho_R + ij\rho_H$,

$$\mathcal{L}(\rho_R + ij\rho_H) = \frac{1}{2}\int \mathrm{d}x^d \left[\partial^i \Pi \partial_i \Pi + 2m^2\Pi^2 + \text{higher terms}\right]; \quad (73)$$

therefore, this Hermitian field Π encodes two massive radial oscillations that are orthogonal, in field space, to the two-dimensional vacuum manifold described in Fig. 11; the corresponding mass is also Hermitian.

All these scalar fields, massless and massive bosons will be completely eaten by a vector field through the Higgs mechanism, once we consider the coupling to hypercomplex QED and local rotations; this will lead to massive pure electrodynamics, without the presence of Higgs massive fields.

4.6 Formation of global topological strings: confirming the Derrick's theorem

Field configurations that define topological defects correspond to domains where the symmetry is left unbroken, i.e. satisfy the constraint $\phi = 0$; we consider for example the case of a four-dimensional space-time (x, y, z, t) as background; for time independent field configurations of the form (1), $\phi = \phi_1 + i\phi_2 + j\phi_3 + ij\phi_4$, such a constraint implies that $\phi_1(x, y, z) = \phi_2(x, y, z) = \phi_3(x, y, z) = \phi_4(x, y, z) = 0$, which define monopoles as possible topological defects. In the same background a conventional complex field of the form $\phi = \phi_1 + i\phi_2$, leads to string defects. Therefore, a hypercomplex field of the form (1) leads to monopoles in a five-dimensional background, and to string defects in a six-dimensional background. However, for a hypercomplex field of the form (4) defined in terms of two real functions, we have again string defects as possible topological defects in a four-dimensional space-time, such as a conventional complex field. These algebraic constraints complemented with a vacuum manifold with nontrivial fundamental group will yield the possibility of formation of string defects in the theories considered in this paper.

In the usual description of $U(1)$-global strings in three spatial dimensions [28–30], the minima lie on a circle with $\psi_0 = \sqrt{\frac{6m^2}{\lambda}} e^{i\alpha}$, where the phase α takes values in $[0, 2\pi]$, and by continuity, within the circle the field must vanish $\psi = 0$; the locus of such points lies on the core of the string defect. Thus, for the case at hand, one can relate each point in the locus to each value of the hyperbolic parameter χ in the interval $(-\infty, +\infty)$. Therefore, by enlarging the usual $U(1)$ vacuum manifold to a cylindrical manifold, the string defect will lie over the new non-compact transversal direction; the formation of this defect is consistent with the fact that a cylindrical manifold is homotopic to S^1; hence, topologically distinct string defects are labeled by the same elements of the fundamental group of S^1, the usual winding number $n \in \mathcal{Z} - \{0\}$.

For the usual $U(1)$-strings with the core aligned with the z-axis, the field asymptotically takes the form

$$\lim_{r\to+\infty} \psi(r, z, \theta) = \rho_0 e^{in\theta}, \quad \rho_0 = \sqrt{\frac{6m^2}{\lambda}}, \tag{74}$$

in cylindrical polar coordinates (r, z, θ); and for the case at hand we have the extended version that incorporates a hyperbolic phase that breaks the translational invariance in the non-compact z-direction of Eq. (74),

$$\lim_{r\to+\infty} \psi(r, z, \theta) = \rho_0 e^{in\theta} e^{jlz}, \quad \rho_0 = \rho_0^R + ij\rho_0^H; \tag{75}$$

where lz is adimensional in natural units; with this condition we identify the asymptotic form of the field with its ground state.

Now, following the usual treatment for string defects, we look for a static exact solution for the equations of motion (35) with the ansatz

$$\psi = \rho_0 f(\rho_0 r) e^{in\theta} e^{jlz}, \quad f(0) = 0, \quad \lim_{r\to+\infty} f(\rho_0 r) = 1; \tag{76}$$

where $\rho_0 r$, and lz are adimensional variables in natural units; the asymptotic limit is required by consistency with the condition (75). With the substitution into the equations of motion (35), these reduce to a non-linear ordinary equation for f;

$$f'' + \frac{1}{\rho_0 r} f' - \frac{n^2}{(\rho_0 r)^2} f - (f^2 - 1) f + \frac{l^2}{\rho_0^2} f = 0, \tag{77}$$

with the exception of the last term f, which comes from the new term $f \partial^2 z e^{jlz}$, all terms correspond to the usual ones in the traditional scheme; hence, the approximate asymptotic solutions remain essentially the same,

$$\begin{aligned} f(\rho_0 r) &\approx C_n r^n + \cdots, \quad r \to 0, \\ f(\rho_0 r) &\approx 1 - \mathcal{O}(r^{-2}), \quad r \to \infty. \end{aligned} \tag{78}$$

Now the gradient of the field is

$$\nabla\psi = \rho_0 \left[f'\hat{r} + in\frac{f}{r}\hat{\theta} + jlf\hat{z} \right] e^{in\theta} e^{jlz}, \tag{79}$$

with a new contribution in the $\hat{z}$-direction; thus the gradient energy density is

$$|\nabla\psi|^2 = \rho_0^2 \left[(f')^2 + n^2\frac{f^2}{r^2} - l^2 f^2 \right]; \tag{80}$$

hence, the energy per unit length is given by

$$2\pi\rho_0^2 \left(n^2 \int^\infty \frac{\mathrm{d}r}{r} - l^2 \int^\infty r\,\mathrm{d}r \right), \tag{81}$$

where we have the usual logarithmical contribution to an infinite energy, and additionally we have a new contribution with a quadratic divergence, which is worse than the logarithmical contribution. Therefore, the energy is infinite, in fact tending to $-\infty$ instead of $+\infty$ as in the usual case; this result is consistent with the Derrick theorem [31], which establishes that there are no finite-energy, time independent solutions, with scalar fields only, that are localized in more than one dimension. However, in the present formulation such a divergence can be cured by considering compensating gauge fields (see Sect. 5.4), such as in the usual $U(1)$ global strings.

5 Hypercomplex electrodynamics: local symmetries

For the usual formulation that describes a charged scalar field coupled to QED, we have the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu - ieA_\mu)\psi|^2 - V(\psi, \overline{\psi}), \tag{82}$$

with two coupling constants e and λ; the hyperbolic rotations can be incorporated as part of the local gauge symmetry of the Lagrangian (82), by considering the expression (4) for the hypercomplex extension of the modulus $\psi \cdot \overline{\psi}$, and the local gauge transformations

$$\psi \to e^{i\theta}e^{j\chi}\psi, \quad A_\mu \to A_\mu + \frac{1}{e}(\partial_\mu\theta - ij\partial_\mu\chi),$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F_{\mu\nu}, \tag{83}$$

where in general the arbitrary real functions depend on the background space-time coordinates, $\theta = \theta(x)$, $\chi = \chi(x)$. Furthermore, Eq. (83) imply that $\overline{A}_\mu \to \overline{A}_\mu + \frac{1}{e}(\partial_\mu\theta - ij\partial_\mu\chi)$, and thus, the condition $A_\mu - \overline{A}_\mu = 0$ is preserved under gauge transformations, in spite of the hypercomplex extension of the fields; hence the vector potential is "Hermitian" in the hypercomplex sense.

The equations of motion and the energy-momentum tensor for the action (82) are,

$$\partial^\mu F_{\mu\nu} = ie(\psi\overline{D_\nu\psi} - \overline{\psi}D_\nu\psi), \tag{84}$$
$$(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu)\psi + \frac{\lambda}{12}(\psi\overline{\psi} - \frac{6m^2}{\lambda})\psi = 0, \tag{85}$$
$$T_{\mu\nu} = D_{(\mu}\psi \cdot \overline{D_{\nu)}\psi} - \frac{1}{2}F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{2}g_{\mu\nu}\mathcal{L}. \tag{86}$$

Similarly in this case, the states with minimal energy are given by $\psi_0\overline{\psi}_0 = \frac{6m^2}{\lambda}$, and the vector potential is pure gauge, $A^0_\mu = \frac{1}{e}(\partial_\mu\theta_0 - ij\partial_\mu\chi_0)$, with θ_0 and χ_0 arbitrary space-time dependent functions; hence, for vacuum fields we must have $\nabla^0\psi^0 \equiv \partial_\mu\psi^0 - ieA^0_\mu\psi^0 = 0$, identically, which will be used implicitly below. Hence, the degenerate vacuum is essentially the manifold described in Fig. 11.

First, we use the parametrization of the fields ψ and ψ_0 given in Eq. (5); the expansions (32), (33), and (34), remain valid in the case at hand since they do not contain the (covariant) derivatives of the fields; one only requires switching on the space-time dependence of the phases $(\chi, \chi_0; \theta, \theta_0)$. Additionally, one must develop the expansion of the first two terms in Eq. (82); expanding around the vacuum requires $\psi \to \psi + \psi_0$, and $A_\mu \to A_\mu + A^0_\mu$, and thus the covariant derivative $(\partial_\mu - ieA_\mu)\psi \to (\partial_\mu - ieA_\mu)\psi - ie(A_\mu\psi_0 + A^0_\mu\psi)$; the Lagrangian (82) reads

$$\begin{aligned}
\mathcal{L}(\psi + \psi_0, A + A_0) &= -\frac{1}{4}F^2_{\mu\nu} + e^2|\psi_0|^2B^2_\mu \\
&+\partial_\mu\psi \cdot \partial^\mu\overline{\psi} + B^\mu\Big\{ie(\overline{\psi}_0\partial_\mu\psi - \psi_0\partial_\mu\overline{\psi}) \\
&+2e|\psi_0|^2(\partial_\mu\theta - ij\partial_\mu\chi) + A^0_\mu(\psi_0\overline{\psi} + \overline{\psi}_0\psi)\Big\} \\
&+i(\partial^\mu\theta - ij\partial^\mu\chi)(\overline{\psi}_0\partial_\mu\psi - \psi_0\partial_\mu\overline{\psi}) \\
&+ieA^\mu_0(\overline{\psi}\partial_\mu\psi - \psi\partial_\mu\overline{\psi}) \\
&+|\psi_0|^2(\partial_\mu\theta - ij\partial_\mu\chi)(\partial^\mu\theta - ij\partial^\mu\chi) \\
&-V(\psi + \psi_0, \overline{\psi} + \overline{\psi_0}) + \text{higher terms},
\end{aligned} \tag{87}$$

where A_μ has been replaced by $A_\mu = B_\mu + \frac{1}{e}(\partial_\mu\theta - ij\partial_\mu\chi)$, and $F_{\mu\nu}$ is expressed now in terms of the new field B_μ. Additionally, we have

$$\partial_\mu\psi \cdot \partial^\mu\overline{\psi} \approx (\gamma^2 - 1)(\partial v^2 + \partial w^2) + 2ij\gamma(\partial w^2 - \partial v^2),$$
$$\overline{\psi}_0\partial_\mu\psi - \psi_0\partial_\mu\overline{\psi} \approx 2i(\gamma^2 + 1)(w_0\partial_\mu v - v_0\partial_u w); \tag{88}$$

Relevant terms in Eq. (32) that lead to quadratic terms in the Lagrangian (87) are given, to second order, by Eqs. (58), and (62), by approaching the circular and hyperbolic phases to first order,

$$-\frac{am^2}{2}\psi\overline{\psi} + \frac{\lambda}{4!}(\overline{\psi}^2_0\psi^2 + \psi^2_0\overline{\psi}^2)$$
$$\approx \begin{cases} -aijP^v_H m^2_R v^2 + \cdots; & (v_0 \neq 0, w_0 = 0) \\ -am^2_R Q^w_R w^2 - aijm^2_R Q^w_H w^2 + \cdots; & (v_0 = 0, w_0 \neq 0) \end{cases}; \tag{89}$$

now, the vanishing requirement of interaction terms of the form $B^\mu \cdot \partial_\mu(\varphi, \overline{\varphi}, \theta, \chi)$ in Eq. (87) yields

$$ij\partial_\mu\Big[2\gamma(w^2_0 - v^2_0)\theta - (\gamma^2 - 1)(w^2_0 + v^2_0)\chi\Big] = 0, \tag{90}$$
$$\partial_\mu\Big[(\gamma^2 - 1)(w^2_0 + v^2_0)\theta + 2\gamma(w^2_0 - v^2_0)\chi$$
$$-(\gamma^2 + 1)(w_0 v - v_0 w)\Big] = 0; \tag{91}$$

Eq. (90) implies that

$$\chi = \frac{2\gamma}{\gamma^2 - 1}\frac{w^2_0 - v^2_0}{w^2_0 + v^2_0}\theta; \tag{92}$$

and thus, Eq. (91) leads to

$$\theta = \frac{(\gamma^4 - 1)(w^2_0 + v^2_0)}{(\gamma^2 - 1)^2(w^2_0 + v^2_0)^2 + 4\gamma^2(w^2_0 - v^2_0)^2}(w_0 v - v_0 w); \tag{93}$$

which corresponds to exploit the freedom of choosing the original gauge field A_μ. We require these expressions when only one of the fields acquires a non-zero expectation value;

$$\theta = \begin{cases} \frac{1-\gamma^2}{1+\gamma^2}\frac{w}{v_0}; & (v_0 \neq 0, w_0 = 0) \\ \frac{\gamma^2-1}{\gamma^2+1}\frac{v}{w_0}; & (v_0 = 0, w_0 \neq 0) \end{cases}; \tag{94}$$

$$\chi = \begin{cases} -\frac{2\gamma}{\gamma^2-1}\theta; & (v_0 \neq 0, w_0 = 0) \\ \frac{2\gamma}{\gamma^2-1}\theta; & (v_0 = 0, w_0 \neq 0) \end{cases}. \tag{95}$$

Thus, using the general expressions (92), and (93), the quadratic terms in the Lagrangian reduce to

$$\mathcal{L}(\psi + \psi_0, A + A_0) = -\frac{1}{4}F^2_{\mu\nu} + e^2|\psi_0|^2B_\mu B^\mu$$
$$+(\gamma^2 - 1)(\partial v^2 + \partial w^2) + 2ij\gamma(\partial w^2 - \partial v^2)$$
$$+(\gamma^2 + 1)^2\frac{(1 - \gamma^2)(w^2_0 + v^2_0) + 2ij\gamma(w^2_0 - v^2_0)}{(\gamma^2 - 1)^2(w^2_0 + v^2_0)^2 + 4\gamma^2(w^2_0 - v^2_0)^2}$$

$$\times\left[w_0^2\partial v^2 - v_0w_0\partial^\mu v\partial_\mu w + v_0^2\partial w^2\right]$$
$$-\left[-\frac{am^2}{2}\psi\overline{\psi} + \frac{\lambda}{4!}(\overline{\psi}_0^2\psi^2 + \psi_0^2\overline{\psi}^2)\right] + \text{higher terms}; \tag{96}$$

in this expression, we have not yet fixed one of the vacuum fields (v_0, w_0), and we have remanent mixed terms of the form $\partial^\mu v \cdot \partial_\mu w$, and $v \cdot w$, which turn out to be proportional to $v_0 \cdot w_0$. Since only one v.e.v., v_0 or w_0 will acquire a non-zero value, such mixed terms will vanish at the end; now we shall study each case separately.

5.1 The case $(v_0 \neq 0, w_0 = 0)$

Using Eqs. (89), the Lagrangian (96) reduces to

$$\mathcal{L}(\psi + \psi_0, A + A_0) = -\frac{1}{4}F_{\mu\nu}^2 + e^2|\psi_0|^2 B_\mu B^\mu$$
$$+(\gamma^2 - 1 - 2ij\gamma)\partial v^2 + aijP_H^v m_R^2 v^2$$
$$+\text{higher terms}; \tag{97}$$

the kinetic and the mass terms for the field w have disappeared, and then w corresponds to a Nambu–Goldstone field. Additionally we have a residual massive field v with a non-zero vacuum expectation value given by Eq. (48); the v.e.v. of this Higgs field determines the (Hermitian) mass of the longitudinal mode of the (Hermitian) vector field B,

$$e^2|\psi_0|^2 = e^2(\gamma^2 - 1 - 2ij\gamma)v_0^2$$
$$= \frac{6e^2}{a\lambda}\left[M_R^B(\gamma) + ijM_H^B(\gamma)\right]m_R^2;$$
$$M_R^B(\gamma) \equiv \frac{1-\gamma^2}{\gamma^2 + 2\gamma - 1}, \quad M_H^B(\gamma) \equiv \frac{2\gamma}{\gamma^2 + 2\gamma - 1}; \tag{98}$$

therefore, the Hermitian vector field B, which in general has the form $B \equiv B_R + ijB_H$, has acquired a Hermitian mass through the Higgs mechanism; hence, one has two real masses for two real fields (B_R, B_H). The flows of the masses defined by these polynomials are shown in Fig. 12; this figure shows the global behavior, and the behavior in the interval $(\gamma_H, -\gamma_H)$. The flows have the same asymptote, the root $\sqrt{2} - 1$ of the polynomial $(\gamma^2 + 2\gamma - 1)$. The real mass of B vanishes at two roots of M_R^B, $\gamma^2 = 1$, the purely hyperbolic limit for the theory; however, these roots are out of the interval $(\gamma_H, -\gamma_H)$.

The figure also shows that the Higgs boson field v may have a mass as small as $\gamma \to \gamma_H$, although strict masslessness is prohibited; hence, in this limit one could expect a nearly massless Higgs boson. In this limit the masses for the vectorial field B will acquire the smallest values,

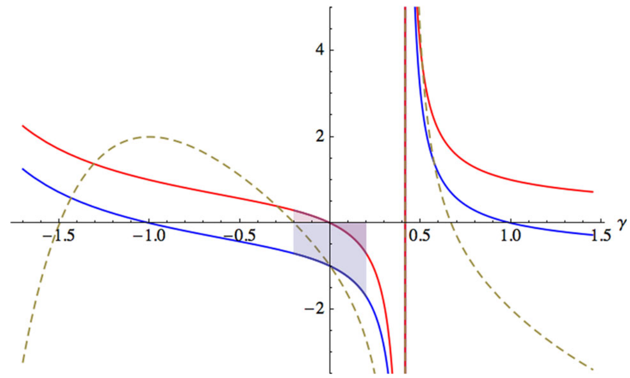

Fig. 12 The *blue curve* represents $M_R^B(\gamma)$, and the *red curve* $M_H^B(\gamma)$; the *dashed curve* represents $P_H^v(\gamma)$ in Eq. (97); the case $\gamma = 0$ reproduces the usual SSB of $U(1)$, with $P_H^v(0) = -1 = M_R^B(0)$, and $M_H^B(0) = 0$

$$\frac{6e^2}{a\lambda}\left[M_R^B(\gamma_H) + ijM_H^B(\gamma_H)\right]m_R^2$$
$$\approx \frac{6e^2}{a\lambda}[-0.7071 + 0.2929ij]\,m_R^2. \tag{99}$$

Similarly in the limit $\gamma \to -\gamma_H$, the figure shows that the fields will acquire the higher masses,

$$\frac{6e^2}{a\lambda}\left[M_R^B(-\gamma_H) + ijM_H^B(-\gamma_H)\right]m_R^2$$
$$\approx \frac{6e^2}{a\lambda}[-1.7071 - .7071ij]\,m_R^2. \tag{100}$$

The limit $\gamma \to \gamma_H$ is the limit of light masses for the fields, and $\gamma \to -\gamma_H$ corresponds to the limit for massive fields; note that the difference between such limits is one mass unit for both, real and hybrid components.

5.2 The case $(v_0 = 0, w_0 \neq 0)$

In this case, the Lagrangian (96) reduces to

$$\mathcal{L}(\psi + \psi_0, A + A_0) = -\frac{1}{4}F_{\mu\nu}^2 + e^2|\psi_0|^2 B_\mu B^\mu$$
$$+(\gamma^2 - 1 + 2ij\gamma)\partial w^2 + am_R^2 Q_R^w w^2 + aijm_R^2 Q_H^w w^2;$$
$$+\text{higher terms}; \tag{101}$$

now the kinetic and the mass terms for the field v have disappeared, and the field v corresponds to a Nambu–Goldstone field. The residual field w is massive in both senses, real and hybrid; the non-zero vacuum expectation value for this field is given by $w_0^2 = \frac{6m_R^2}{a\lambda(1-\gamma^2+2\gamma)}$ (see Sect. 4.3), and determines the masses for the vectorial field B shown in Fig. 13. This figure is basically the mirrored image with respect to the "y"-axis of Fig. 12; the qualitative and quantitative aspects of Eqs. (99), and (100) remain valid, and one only requires interchanging the roles of the limits $\gamma_H \leftrightarrow -\gamma_H$. The difference with respect to the Lagrangian (97) is that the Higgs

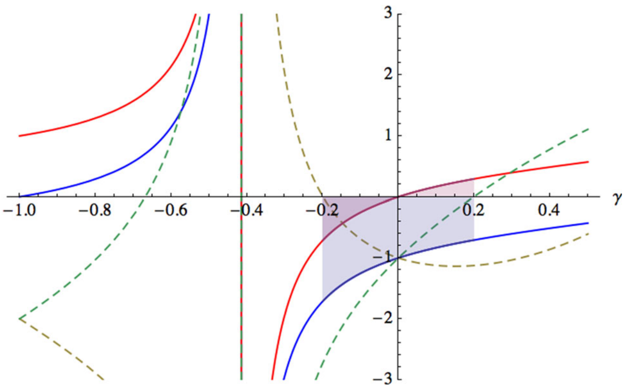

Fig. 13 The mirrored image of Fig. 12

field w appears with real and hybrid masses, and their flows are shown as dashed curves; this behavior was discussed in Fig. 9, and hence, in the limit $\gamma \to \gamma_H$, the Higgs field w will have a light real mass and a hybrid mass with its maximum value, and conversely in the limit $\gamma \to -\gamma_H$.

5.3 Polar parametrization for the fields: purely massive electrodynamics, no massive Higgs bosons

Along the lines followed in Sect. 4.5, we consider the polar decomposition for the fields with radial and circular modes; in the expression of the form $(\rho_R + ij\rho_H)e^{i\xi}e^{j\eta}$, for the dynamical field we can consider for generality that the four variables ρ_R, ρ_H, ξ, and η, are space-time coordinates dependent, and at the end we shall consider that two of them must be constants. Hence, one has the following expressions:

$$\begin{aligned}
&\partial_\mu\psi \cdot \partial^\mu\overline{\psi} = (\partial\rho_R + ij\partial\rho_H)^2 - (\rho_R + ij\rho_H)^2(i\partial\xi + i\partial\eta)^2,\\
&\overline{\psi}\partial_\mu\psi - \psi\partial_\mu\overline{\psi} = 2(\rho_R + ij\rho_H)^2(i\partial\xi + i\partial\eta),\\
&V(\psi + \psi_0, \overline{\psi} + \overline{\psi}_0) = m^2\rho^2 + \text{higher terms}, \qquad (102)
\end{aligned}$$

and then

$$\begin{aligned}
&|(\partial_\mu - ieA_\mu)\psi|^2 = (\partial\rho_R + ij\partial\rho_H)^2 + e(\rho_R + ij\rho_H)^2\\
&\quad\times\Big[eB^\mu B_\mu + 2B^\mu\partial_\mu[\theta - \xi + ij(\eta - \chi)]\\
&\quad -\frac{1}{e}(i\partial\xi + i\partial\eta)^2 + \frac{1}{e}\partial^\mu(\theta - ij\chi)\\
&\quad \cdot\partial_\mu[\theta - 2\xi + ij(2\eta - \chi)]\Big], \qquad (103)
\end{aligned}$$

where $A_\mu = B_\mu + \frac{1}{e}(\partial_\mu\theta - ij\partial_\mu\chi)$; fluctuations around the vacuum require one to rewrite the dynamical field as $\psi \to (\rho_R + \rho_R^0 + ij(\rho_H + \rho_H^0))e^{i\xi}e^{j\eta}$, which inserted into the above expressions leads to

$$\begin{aligned}
&\mathcal{L}(\psi + \psi_0, A + A_0)\\
&\quad = -\frac{1}{4}F^2 + e^2|\psi_0|^2B^2 + (\partial\rho_R + ij\partial\rho_H)^2\\
&\qquad -|\psi_0|^2(i\partial\xi + i\partial\eta)^2\\
&\qquad +2e|\psi_0|^2B^\mu\partial_\mu[\theta - \xi + ij(\eta - \chi)]\\
&\qquad +|\psi_0|^2\partial^\mu(\theta - ij\chi)\cdot\partial_\mu[\theta - 2\xi + ij(2\eta - \chi)]\\
&\qquad +am^2(\rho_R + ij\rho_H)^2 + \text{higher terms}. \qquad (104)
\end{aligned}$$

The case $\gamma^2 = 1$: hyperbolic electrodynamics
Along the lines followed in Sect. 4.5 for this case, we consider first Eq. (66), with $\eta = constant$; the vanishing of the interaction terms of the form $B^\mu \cdot \partial_\mu(\xi, \eta, \theta, \chi)$ in Eq. (104) requires the identification $\theta(x) = \xi(x)$, and the fixing of the hyperbolic parameter $\chi = constant$;

$$\begin{aligned}
&\mathcal{L}(\psi + \psi_0, A + A_0) = -\frac{1}{4}F^2 - ij\frac{6e^2m_H^2}{a\lambda}B^2\\
&\quad +2\gamma ij[(\partial\rho)^2 + aijm_H^2\rho^2] + \text{higher terms}. \qquad (105)
\end{aligned}$$

Hence, the field $\xi(x)$ has disappeared and represents a Nambu–Goldstone boson in the two-dimensional valley. Therefore, as opposed to the global SSB scenario described in Eq. (67), the field ξ has not survived. Furthermore, in an orthogonal valley direction, we have a massive mode ρ, which has survived the gauging of the global SSB to the local symmetry.

Similarly, in the case with a field of the form (66), with $\xi = constant$, the quadratic terms in the Lagrangian have essentially the same form shown in Eq. (105), where the interaction terms, and the terms involving the fields $(\chi(x), \eta(x))$ vanish by fixing now $\theta = constant$, and identifying $\chi(x) = \eta(x)$.

Now, using the parametrization (68), which encodes all fluctuations in the phases, the identifications $\theta(x) = \xi(x)$, and $\chi(x) = \eta(x)$ lead to the vanishing of the interaction terms in Eq. (104), and all terms involving the scalar fields; the Lagrangian reduces to second order to a massive pure electrodynamics,

$$L = -\frac{1}{4}F^2 - ij\frac{6e^2m_H^2}{a\lambda}B^2 + \text{higher terms}; \qquad (106)$$

thus, the gauge vector boson has eaten the two Nambu–Goldstone bosons (ξ, η) and has acquired a mass; the field excitations in the valley directions described in the Lagrangian (69) have not survived the gauging of a global SSB to a local symmetry, and there are no scalar Higgs fields.

In the case of the conventional scalar electrodynamics one has the vector gauge field A, a complex field $\phi = \phi_1 + i\phi_2$, with two real scalar fields, and the S^1 vacuum manifold with only one generator; thus, after the spontaneous symmetry breaking, only one scalar field can be eaten through the Higgs mechanism, leaving a massive vector field and additionally a

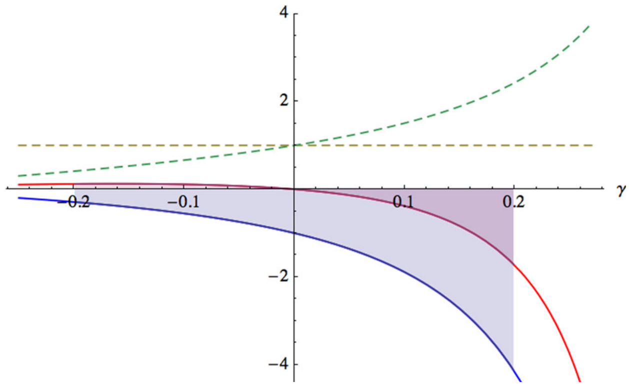

Fig. 14 The constant real mass is represented by the *horizontal dashed line*; the running hybrid mass is represented by the *dashed curve*

massive scalar field, as opposed to the massive pure electrodynamics at hand, with a hypercomplex field with two real fields, and a vacuum manifold with two generators.

The case $\gamma^2 \neq 1$: hyperbolic deformation of QED

For this scenario of running parameters we shall consider two extremal cases; the first case corresponds to two massive modes with oscillations orthogonal to the two-dimensional valley, and the second case with two redundant modes oscillating on the valley.

For the first case we consider the parametrization given in Eq. (71), and the Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F^2 + e^2|\psi_0|^2 B^2 + (\partial\rho_R + ij\partial\rho_H)^2 \\ +am^2(\rho_R + ij\rho_H)^2 + \text{higher terms}; \tag{107}$$

where

$$m^2 = \begin{cases} m_R^2\left(1 + ij\frac{\gamma^2-2\gamma-1}{\gamma^2+2\gamma-1}\right); & (v_0 \neq 0, w_0 = 0) \\ m_R^2\left(1 + ij\frac{\gamma^2+2\gamma-1}{\gamma^2-2\gamma-1}\right); & (v_0 = 0, w_0 \neq 0) \end{cases}; \tag{108}$$

therefore, we have a Hermitian scalar field with a Hermitian mass, which has a constant real mass, and a running hybrid mass; for the first case of the above equation the mass of the vector field B, is described in Sect. 5.1 and shown in Fig. 12. In this case, the difference is the presence of a Hermitian Higgs field with different masses; Fig. 14 shows again the masses of the field B shown in Fig. 12, but now with the Higgs masses given in Eq. (108). The figure shows that in this case the limit $\gamma \to \gamma_H$ is not a limit for light Higgs fields; however, the limit $\gamma \to -\gamma_H$ corresponds again to a massive fields limit. The hybrid component of the mass is running in the interval $m_H^2 \in (m_H^2(\gamma_H), m_H^2(-\gamma_H)) \approx (0.4142, 2.4142)m_R^2$.

Furthermore, the second case described in Eq. (108) can be obtained from the first one by the transformation $\gamma \to -\gamma$, and the corresponding figure is basically the mirrored image with respect to the "y"-axis of Fig. 14.

The case with two scalar redundant modes can be developed along the ideas behind Eq. (106), leading to the Lagrangian

$$L = -\frac{1}{4}F^2 + e^2|\psi_0|^2 B^2 + \text{higher terms}; \tag{109}$$

the only difference is the Hermitian mass for the vector field B; depending on the vacuum expectation values (v_0, w_0) such a running mass is described by Fig. 12, or the mirrored figure, Fig. 13; in both cases there will be no dashed curves, since the scalar Higgs fields have strictly disappeared; we have again a purely massive electrodynamics, without any clues of scalar fields.

5.4 Local topological strings: Aharanov–Bohm-like defects

The geometrical and topological description of the formation of global string defects given in Sect. 4.6 is valid in essence for the local case, adding the asymptotic form for the gauge field, and the gradients of the fields,

$$\lim_{r\to+\infty} \psi(r,z,\theta) = \rho_0 e^{in\theta} e^{jlz}, \\ \lim_{r\to+\infty} D_A\psi = 0, \quad n \in \mathcal{Z} - \{0\}; \\ \lim_{r\to+\infty} \mathbf{A}(r,z,\theta) = \frac{1}{e}\nabla(n\theta - ijlz) = \frac{1}{e}\left(\frac{n}{r}\hat{\theta} - ijl\hat{z}\right), \\ \lim_{r\to+\infty} F_{\mu\nu} = 0; \tag{110}$$

these asymptotic expressions correspond to the boundary conditions for strings solutions of nontrivial windings of a circle onto the vacuum manifold. Therefore, with these conditions, the magnetic flux passing through a closed surface S is given by

$$\int_S \mathbf{B}\cdot\mathbf{dS} = \oint_C \mathbf{A}\cdot\mathbf{dC} = -\frac{2\pi n}{e}, \quad \mathbf{dC} = r\hat{\theta}d\theta; \tag{111}$$

where C is an infinitely large loop at spatial infinity, and we have used the Stokes theorem; at this region, the vector $\hat{\theta}$ points tangentially to the loop, and $\hat{z}$ points orthogonally in the cylindrical direction. Therefore, the conventional quantized magnetic flux for string defects remains intact.

Now the ansatz includes Eq. (76) for the field ψ, and the following expression for the gauge field:

$$\mathbf{A}(r,z,\theta) = \frac{i}{e}f_A(r)e^{in\theta}e^{jlz}\nabla(e^{-in\theta}\cdot e^{-jlz}) \\ = \frac{1}{e}f_A(r)\left(\frac{n}{r}\widehat{\theta} - ijl\widehat{z}\right), \\ f_A(0) = 0, \quad f_A(\infty) = 1; \tag{112}$$

in consistency with the asymptotic limits (110). This potential results in a magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{f'_A}{e}\left(\frac{n}{r}\widehat{z} + ijl\widehat{\theta}\right), \tag{113}$$

where we have a new hybrid contribution in the circular direction; note that $\overline{\mathbf{B}} = \mathbf{B}$.

Therefore, from Eqs. (76), (112), and (113), the equations of motion (84) reduce to

$$-\frac{n}{e}\frac{d}{\mathrm{d}r}\left(\frac{f'_A}{r}\right)\widehat{\theta} + \frac{ijl}{e}\frac{1}{r}\frac{d}{\mathrm{d}r}(rf'_A)\widehat{z} = 2e\rho_0^2 f^2(1-f_A)\cdot\left(\frac{n}{r}\widehat{\theta} - ijl\widehat{z}\right), \tag{114}$$

where the new contribution correspond to hybrid terms in the $\widehat{z}$-direction on both sides; these equations reduce explicitly to

$$\widehat{\theta} : rf''_A - f'_A + 2e^2\rho_0^2 r(1-f_A)f^2 = 0, \tag{115}$$

$$\widehat{z} : rf''_A + f'_A + 2e^2\rho_0^2 r(1-f_A)f^2 = 0. \tag{116}$$

Additionally, Eq. (85) reduces to

$$\frac{1}{r}\frac{d}{\mathrm{d}r}(rf') - f(1-f_A)^2\left(\frac{n^2}{r^2} - \underbrace{l^2}\right) + \frac{m^2}{2}(1-f^2)f = 0, \tag{117}$$

where we have underbraced the new contribution coming from the hyperbolic phase e^{jlz}. Furthermore, Eq. (115) is exactly the same equation obtained in the usual formulation for the $U(1)$ local strings; this equation together with the corresponding equation of the form (117), have no closed-form solutions. Hence the asymptotic analysis is used for large r, and close to the vortex core. However, the presence of Eq. (116), distinctive of this hypercomplex formulation, has a dramatic effect on the possible solutions, enforcing the vacuum configuration for f_A in full space, except for $r = 0$;

$$f_A(0) = 0, \qquad f_A(r) = 1, \qquad r \in (0, +\infty), \tag{118}$$

thus, the potential is "pure gauge",

$$\mathbf{A}(0) = 0; \qquad \mathbf{B}(0) = 0, \quad r = 0, \quad \text{in the vortex core}, \tag{119}$$

$$\mathbf{A} = \frac{1}{e}\left(\frac{n}{r}\widehat{\theta} - ijl\widehat{z}\right); \qquad \mathbf{B}(r) = 0, \qquad r \in (0, +\infty), \quad \text{elsewhere}; \tag{120}$$

this closed-form solution shows two representative features of a topological defect, namely the field configurations where the symmetry is left unbroken, and the configurations at the vacuum, where the symmetry will be spontaneously broken; note from Eq. (120) that the potential $\mathbf{A}$ cannot vanish at the vortex core, due to the restriction $n \neq 0$. The presence of "pure gauge" potentials in spaces with nontrivial topology, invokes immediately the Aharanov–Bohm effect [32]: a narrow, infinite length solenoid is added to the two-slit experiment for electrons; topologically the solenoid is a string-like defect, and thus the phase shift on the electron wave function is a topological effect determined by the magnetic flux inside the solenoid. Outside, there is no magnetic field, with non-zero potential. Since the original experimental confirmation [33], the Aharanov–Bohm effect has received considerable study, from theoretical generalizations, to a variety of experimental realizations; for example, the appearance of the electronic interference phenomenon in carbon nanotubes suggests that the Aharanov–Bohm effect is relevant even at the microscopic scale [34]. With this perspective, we establish now the analogy with the case at hand.

In the analogy, the infinite string core corresponds to the Aharanov–Bohm solenoid; the azimuthal component of $\mathbf{A}$ in Eq. (120) falls off like $1/r$, with the distance from the core, such as in the Aharanov–Bohm effect. Additionally the new contribution in the $\hat{z}$-direction is constant everywhere; this new contribution does not change the quantization condition of the magnetic flux in (111). Now an important difference; in the core of the string defect the magnetic field vanishes, as opposed to the Aharanov–Bohm solenoid, inside which the $\mathbf{B}$ field is non-zero; the string defect at hand is actually a 'pure gauge" phenomenon. Furthermore, in the background, one has the same topological feature, namely, the existence of a non-simply connected space, and thus the winding number around the loop is observable in the Aharanov–Bohm effect; in the analogy, this fact represents a phenomenological possibility for the string defects at hand.

Let us see how these "pure gauge" potentials are able to make finite the energy of the defect, playing the role of compensating fields for the divergent effect of the scalar fields.

The substitution of the solution (118) into Eq. (117) leads to a simplified equation,

$$\frac{1}{r}\frac{d}{\mathrm{d}r}(rf') + \frac{m^2}{2}(1-f^2)f = 0; \tag{121}$$

an immediate solution is $f = 1$, and thus, the field ψ settles down to its vacuum configuration in the full space, except for the core. Note that this solution for f does not solve Eq. (77) for global strings. In this case, all components of the energy-momentum tensor (86) vanish trivially, and hence are finite. Therefore, all classical Maxwell physical observables vanish, and there is no classical experiment that allows us to detect the string defect by its electrodynamical effects. However, the loop integral (111) is a gauge invariant quantity and, according to the Aharanov–Bohm effect, is detectable by quantum interference.

5.5 There are no other solutions for local strings

For the usual description of local string defects that involves only Eqs. (115) and (117), the asymptotic solutions are well known [30]; hence, close to the vortex core, one has

$$f_A \sim r^2, \quad f \sim r^n, \quad r \to 0; \tag{122}$$

and for large r,

$$1 - f_A \sim e^{-mr}, \quad 1 - f \sim e^{-\beta r}, \quad \beta = \frac{\lambda}{e^2}, \quad r \to \infty; \tag{123}$$

physically, these expressions lead to an energy density more localized that for the global strings, and show that the asymptotic behavior of f is controlled by the gauge field contribution f_A. These expressions are not valid in the present treatment, since f_A, has been relaxed to its vacuum configuration in the full interval $r \in (0, +\infty)$ due to Eq. (116); we have only at hand Eq. (121), which is fully independent on f_A.

Other closed-form solutions for Eqs. (121) are not known; however, one can obtain asymptotic solutions for large r and close to the core using linearized versions; for large r the background field is $f = 1$, and close to the vortex $f = 0$;

$$r\triangle f'' + \triangle f' - m^2 r\triangle f = 0, \quad r \to \infty,$$
$$\triangle f = AJ_0(imr) + BN_0(-imr), \tag{124}$$

and

$$r\triangle f'' + \triangle f' + \frac{m^2}{2} r\triangle f = 0, \quad r \to 0,$$
$$\triangle f = AJ_0\left(\frac{m}{\sqrt{2}}r\right) + BN_0\left(\frac{m}{\sqrt{2}}r\right), \tag{125}$$

where J_0 is the zeroth-order Bessel function, N_0 the zeroth-order Neumann function, and A and B are adimensional constants. However, in Eq. (125), the condition $A = 0$ is enforced for ensuring a single-valued function at $r = 0$; thus, the energy density is given essentially by the expression

$$T_{00} = -\frac{1}{2}(\triangle f')^2 = -\frac{B^2 m^2}{4}\left[N_1\left(\frac{m}{\sqrt{2}}r\right)\right]^2, \tag{126}$$

where N_1 is the first-order Neumann function; thus, the energy diverges, similarly to the case of global strings. Since local strings have their energy confined mainly close to the core, such a solution is not physically meaningful; in this case the gauge field f_A has disappeared as a compensating field of the divergent effects of the scalar field. Therefore, the Aharonov–Bohm like defects discussed early, are the only field configurations with finite energy.

6 Concluding remarks

6.1 Cosmological implications

The results obtained can be interpreted in various senses; in the inflationary cosmological context, the topological defects are essential in the spontaneous symmetry breaking based phase transitions; if a phase transition occurs, then the topological defects may be generated provided that the vacuum manifold has a nontrivial topology. Since the dramatic effect of the substitution of the circular rotations by the hyperbolic rotations is the trivialization of the vacuum manifold from the homotopic point of view (Sect. 4.1), the insignificant observational support of the existence of cosmic string defects leads to the possibility that the vacuum can manifest a non-compact topology in a certain phase of the early universe. This in its turn has various implications; for example it suggests that the GUT philosophy must be extended by incorporating non-compact gauge groups. Furthermore, the proliferation of cosmic defects is an essential feature of certain GUT's; *inflation* was originally proposed as a form of explaining the observational evidence against such a proliferation. In the light of the present results, one can alternatively postulate the hyperbolic symmetry as an essential symmetry at some period of the early universe; thus, the inflationary scenarios can be modified drastically by the presence of the new symmetry. More explorations are mandatory along these ideas.

6.2 On Aharonov–Bohm strings

It has been shown that the dominant interaction between matter and cosmic strings is through an Aharonov–Bohm interaction [35]; outside the tiny region of the inner core, the field strengths vanish, but one has non-vanishing potentials in the outer region, and hence a scattering mechanism of the Aharonov–Bohm type will work at the outer region. The scattering cross sections and production rates do no go to zero as the geometrical size of the string vanishes. One can realize that the results obtained in the present approach are consistent with those results, but they suggest that the Aharonov–Bohm interaction is the only interaction between matter and cosmic strings; the field configurations for the strings found in Sect. 5.4 do not distinguish between inner core and outer region, since the potentials are pure gauge as close to the core as one wishes.

The Aharonov–Bohm strings have appeared previously in the study of discrete gauge symmetries [36–39]; specifically in the effective Lagrangian description of Z_k discrete gauge theory, this type of strings have confined magnetic flux with a $1/k$ unit of fundamental magnetic charge; due to this charge the Aharonov–Bohm string can interact with matter. Recently, certain cosmological constraints have been imposed on these theories, by studying the radiation of standard model particles from these strings [40]. As opposed to these discrete gauge symmetry approaches, the Aharonov–Bohm-like strings have been obtained in the approach at hand by incorporating a (non-compact) continuous symmetry, the hyperbolic rotations; it may be interesting to study phenomenological models that incorporate both, discrete, and non-compact gauge symmetries, following the ideas described in [40].

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References

1. M.H. Emam, Five-dimensional 2-branes and the universal hypermultiplet. Nucl. Phys. B **817**, 208 (2009). arXiv:hep-th/0701060
2. M.H. Emam, BPS one-branes in five-dimensions. arXiv:1301.7338 [hep-th]
3. M.H. Emam, Phys. Rev. D **74**, 125004 (2006)
4. M.H. Emam, Phys. Rev. D **79**, 085017 (2009)
5. M.H. Emam, Phys. Rev. D **71**, 125020 (2005)
6. M.H. Emam, Phys. Rev. D **84**, 045016 (2011)
7. G.W. Gibbons, M.B. Green, M.J. Perry, Instantons and seven-branes in type IIB superstring theory. Phys. Lett. B **370**, 37 (1996). arXiv:hep-th/9511080
8. V. Cortes, C. Mayer, T. Mohaupt, F. Saueressing, Special geometry of Euclidean supersymmetry. 1. Vector multiplets. JHEP **0403**, 028 (2004). arXiv:hep-th/0312001
9. V. Cortes, C. Mayer, T. Mohaupt, F. Saueressing, Special geometry of Euclidean supersymmetry II: hypermultiplets and the c-map. JHEP **0506**, 025 (2005). arXiv:hep-th/0503094
10. S. Ulrych, Gravitoelectromagnetism in a complex Clifford algebra. Phys. Lett. B **633**, 631 (2006). arXiv:gr-qc/0602018
11. S. Ulrych, Relativistic quantum physics with hyperbolic numbers. Phys. Lett. B **625**, 313 (2005). arXiv:hep-th/9904170
12. S. Ulrych, The Poincaré mass operator in terms of a hyperbolic algebra. Phys. Lett. B **612**, 89 (2005). arXiv:hep-th/0510198
13. J. Hucks, Hyperbolic complex structures in physics. J. Math. Phys. **34**, 5986 (1993)
14. S. Ulrych, Symmetries in the hyperbolic Hilbert space. Phys. Lett. B **618**, 233 (2005). arXiv:hep-th/0510199
15. J. Pati, A. Salam, Unified lepton–hadron symmetry and a gauge theory of the basic interactions. Phys. Rev. D **8**, 1240 (1973)
16. S. Ulrych, Considerations on the hyperbolic complex Klein–Gordon equation. J. Math. Phys. **51**, 063510 (2010). arXiv:1006.5182 [math-ph]
17. L. Faddeev, G. Korchemsky, High energy QCD as a completely integrable model. Phys. Lett. B **342**, 311 (1995). arXiv:hep-th/9404173
18. M. Niedermaier, E. Seiler, Nonamenability and spontaneous symmetry breaking: the hyperbolic spin chain. Annales H. Poincaré **6**, 1025 (2005). arXiv:hep-th/0312293
19. T. Morozumi, S. Nojiri, An analysis of noncompact nonlinear sigma models. Prog. Theor. Phys. **75**, 677 (1986)
20. J.W. van Holten, Quantum non-compact sigma models. J. Math. Phys. **28**, 1420 (1987)
21. S. Mandelstam, Vortices and quark confinement in non-Abelian gauge theories. Phys. Rep. C **23**, 245 (1976)
22. G. t Hooft, Topology of the gauge condition and new confinement phases in non-Abelian gauge theories. Nucl. Phys. B **190**, 455 (1981)
23. P. Faccioli, E. Shuryak, QCD topology at finite temperature: statistical mechanics of self-dual dyons. Phys. Rev. D **87**, 074009 (2013). arXiv:1301.2523 [hep-ph]
24. Y.M. Cho, D.G. Pak, P.M. Zhang, L.P. Zou, Weyl symmetric structure of QCD vacuum. Phys. Rev. D **86**, 045025 (2012). arXiv:hep-th/12045970
25. L.P. Zou, P.M. Zhang, D.G. Pak, Knot topology in QCD. Phys. Lett. B **726**, 436 (2013). arXiv:1302.3527 [hep-th]
26. I. Bengtsson, J. Hallin, SL(2, R) Yang–Mills theory on a circle. Mod. Phys. Lett. A. **9**, 3245 (1994). arXiv:hep-th/9407035
27. C.T. Hill, Is the Higgs boson associated with Coleman–Weinberg dynamical symmetry breaking? Phys. Rev. D **89**, 073003 (2014). arXiv:1401.4185 [hep-ph]
28. T.W.B. Kibble, Classification of topological defects and their relevance in cosmology and elsewhere, in *Topological Defects and the Non-equilibrium Dynamics of Symmetry Breaking Phase Transitions*, ed. by Y.M. Bunkov, et al. (Kluwer Academic Publishers, Berlin, 2000)
29. H. Nielsen, P. Olesen, Vortex-line models for dual strings. Nucl. Phys. B **61**, 45 (1973)
30. A. Vilenkin, E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 1994)
31. G.H. Derrick, Comments on nonlinear wave equations as models for elementary particles. J. Math. Phys. **5**, 1252 (1964)
32. Y. Aharonov, D. Bohm, Significance of electromagnetic potentials in the quantum theory. Phys. Rev. **115**, 485 (1959)
33. R.G. Chambers, Shift of an electron interference pattern by enclosed magnetic flux. Phys. Rev. Lett. **5**, 3 (1960)
34. J. Cao, Q. Wang, M. Rolandi, H. Dai, Aharonov–Bohm interference and beating in single-walled Carbon-nanotube interferometers. Phys. Rev. Lett. **93**, 216803 (2004). arXiv:cond-mat/0407185
35. M.G. Alford, F. Wilczek, Aharonov–Bohm interaction of cosmic strings with matter. Phys. Rev. Lett. **62**, 1071 (1989)
36. L.M. Krauss, F. Wilczek, Discrete gauge symmetry in continuum theories. Phys. Rev. Lett. **62**, 1221 (1989)
37. M.G. Alford, J. March-Russell, F. Wilczek, Discrete quantum hair on black holes and the nonAbelian Aharonov–Bohm effect. Nucl. Phys. B **337**, 695 (1990)
38. J. Preskill, L.M. Krauss, Local discrete symmetry and quantum mechanical hair. Nucl. Phys. B **341**, 50 (1990)
39. T. Banks, N. Seiberg, Symmetries and strings in field theory and gravity. Phys. Rev. D. **83**, 084019 (2011). arXiv:1011.5120 [hep-th]
40. Y. Ookouchi, Discrete gauge symmetry and Aharonov–Bohm radiation in string theory. JHEP **1401**, 049 (2014). arXiv:1310.4026 [hep-th]